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Abstract

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D. M. TOLMAZIN

ON THE THEORY OF CURRENTS IN SEA STRAITS

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Consider motion in a stratified strait of rectangular cross-section, caused by wind, a longitudinal difference in level, and a density gradient. Let the x-axis be directed along the undisturbed sea level toward the sea of lower density, and the z-axis vertically downward. To analyze the distribution of currents and water density in the xoz plane, we consider the equations of motion and mass diffusion in the steady state:

$$\mu u'' = p_x; \tag{1}$$

$$p' = g\rho; \tag{2}$$

$$u\rho_x = k\rho'', \tag{3}$$

where u, p, and ρ are the longitudinal component of velocity, pressure, and water density; μ and k are the coefficients of turbulent viscosity and diffusion, assumed constant in the problem; primes denote differentiation with respect to z, and subscripts x, with respect to x.

Integrating (1) from the sea surface $-\zeta$ to z, and taking (2) into account, after transformations we obtain:

$$u = u^* + z\left(\frac{\partial u}{\partial z}\right)_{z=-\zeta} + \frac{g\rho^* z^2}{2\mu}\zeta_x + \frac{g}{\mu}\frac{\partial}{\partial x}\int_{-\zeta}^{z}\int\int \rho\,(dz)^3, \tag{4}$$

where u^* and ρ^* are the velocity and density at the surface.

Introduce the auxiliary function Q:

$$Q = \int_0^x \left[\mu u^* + \mu z\left(\frac{\partial u}{\partial z}\right)_{z=-\zeta} + \frac{g\rho^* z^2}{2}\zeta_x\right]dx + g\int_{-\zeta}^{z}\int\int \rho\,(dz)^3, \tag{5}$$

related to the velocity and density by the relations

$$u = \frac{1}{\mu} Q_x; \tag{6}$$

$$\rho = \frac{1}{g} Q'''. \tag{7}$$

Substituting (6), (7) into (3), we obtain the basic equation of the problem:

$$Q_x Q_x''' = \mu k Q^{\mathrm{V}}. \tag{8}$$

The boundary conditions take into account the tangential wind stress at the surface T and no slip at the bottom:

$$\mu \left(\frac{\partial u}{\partial z}\right)_{z=-\zeta} = Q_x'(x, \zeta) = -T; \tag{9}$$

$$u\Big|_{z=h} = Q_x(x, h) = 0; \tag{10}$$

$$\frac{\partial \rho}{\partial z}\bigg|_{z=-\zeta} = Q^{\mathrm{IV}}(x, \zeta) = 0; \qquad \frac{\partial \rho}{\partial z} = Q^{\mathrm{IV}} = 0. \tag{11}$$

Conditions (11) were introduced in (1).
The missing fifth condition with respect to z can be obtained from the equation of continuity for a prescribed elevation of the level $\zeta(x)$:

$$\frac{\partial}{\partial x} \int_{-\zeta}^{h} u \, dz = \frac{\partial}{\partial x} \int_{-\zeta}^{h} Q_x \, dz = 0. \tag{12}$$

Let us pass to dimensionless variables, setting

$$x = l\xi, \qquad z = H_0 \bar{z}, \qquad Q = Q_0 \bar{Q}, \qquad h = H_0 \bar{h}, \qquad \rho = \rho_0 \bar{\rho}, \tag{13}$$

where l is the length of the strait, H_0 is the maximum value of the strait depth h, and ρ_0 is the greatest value of ρ. Expressing Q_0 through the defining parameters $Q_0 = g\rho_0 H_0^3$, we obtain the equation in dimensionless form

$$\varepsilon \bar{Q}^{\mathrm{V}} - \bar{Q}_\xi \bar{Q}_\xi'' = 0, \tag{14}$$

where $\varepsilon = \mu k l^3 / g\rho_0 H_0^5$ is a dimensionless small parameter.

We seek the solution $\bar{Q}$ in the form

$$\bar{Q} = \varepsilon M(\xi) f(\eta), \tag{15}$$

where $h = \bar{z}\xi^{\alpha} m^{-1}$, and α and m are as yet unknown parameters determined by the geometry of the strait. Requiring that the relation $0 \leq \eta \leq 1$ be satisfied, we must prescribe the law of variation of the strait depth

$$\bar{h} = m\xi^{-\alpha}. \tag{16}$$

It is evident that real values of $\bar{h}$ and η will occur under the condition that ξ varies in the region $\xi_1 \leq \xi \leq \xi_2$, with necessarily $\xi_1 > 0$. The possibility of varying the parameters α, m, and the dimensionless coordinate of the beginning of the strait ξ_1 makes it possible to take sufficiently accurate account, in the problem, of the general bottom slope in a number of straits.

It is not difficult to verify that in (14) one of the variables can be eliminated only under the condition

$$M(\xi) = \xi^{2\alpha+2}. \tag{17}$$

Substituting (15), taking (17) into account, into (14), we have:

$$f^{\mathrm{V}} = m^2\alpha^2\eta^2 f' f^{\mathrm{IV}} + a\eta f^{\mathrm{IV}} f + b\eta f''' f' + c f''' f, \tag{18}$$

with the conditions

$$\frac{f'(0)}{\varepsilon} = \frac{mt}{\varepsilon^2(3\alpha+2)\xi^{3\alpha+1}} = D_1 \text{ (const)}; \qquad f'(1) + sf(1) = 0; \tag{19}$$

$$f^{\mathrm{IV}}(1) = f^{\mathrm{IV}}(0) = 0; \qquad \int_0^1 [\eta f' + sf]\, d\eta = 0.$$

Here the following notation has been introduced: $a = m^2\alpha(2\alpha+2)$, $b = m^2\alpha(5\alpha+2)$, $c = m^2(2\alpha+2)(5\alpha+2)$, $s = (2\alpha+2)/\alpha$.

In order for the problem to be completely determined, it is necessary to prescribe one more condition:

$$\frac{f'''(0)}{\varepsilon} = \frac{m^3 p(\xi, 0)}{\varepsilon^2 \rho_0 \xi^{5\alpha+2}} = D_2. \tag{20}$$

From conditions (19) and (20) it follows that, under our assumption about the form of $\overline{Q}$ in (15), it is necessary to impose certain restrictions on the longitudinal distribution of the wind and of the density at the surface:

$$\tau = \tau_0 \xi^{3a+1}; \qquad \rho(\xi, 0) = \rho_0 \xi^{5a+2}. \tag{21}$$

Let us construct the solution of problem (18) and (19), expanding f in a series in powers of the small parameter ε. Put

$$f = \varepsilon \sum_{n=0}^{\infty} \varepsilon^n f_n. \tag{22}$$

Substituting (22) into (18) and equating coefficients at equal powers of ε, we find

$$f_0 = A_0 + \sum_{i=1}^{4} A_i \frac{\eta^i}{i!}; \tag{23}$$

$$f_1 = B_0 + A_3 \sum_{i=1}^{8} B_i \frac{\eta^i}{i!}, \tag{24}$$

where

$$A_1 = D_1; \qquad A_3 = D_2;$$

$$A_0 = \frac{1}{4s} \left[\frac{s+3}{12} D_2 - (s+1) D_1 \right];$$

$$A_2 = -\frac{3}{2(s+2)} \left[(s+1) D_1 + \frac{s+3}{4} D_2 \right];$$

$$B_0 = -\frac{A_3}{2s} \sum_{i=5}^{8} A_{i-5} \frac{(i+s)(i-2)}{(i+1)!} [(i-5)b + c];$$

$$B_2 = -\frac{3}{s+2} \sum_{i=5}^{8} A_{i-5} \frac{i(i+5)}{(i+1)!} [(i-5)b + c];$$

$$B_{i+5} = \frac{ib + c}{(i+5)!} A_i;$$

$$B_1 = B_3 = B_4 = 0.$$

Fig. 1. Computed (1) and observed (2) distribution of density at the point $\xi = 1$ of the Bosporus Strait

It is not difficult to find the subsequent approximations as well, but, as the calculation shows, a sufficient degree of accuracy is provided by the first two approximations.

The theory described models well the density and current fields in straits with a density-driven water exchange: the Bosporus, the Dardanelles, Gibraltar, and Bab el-Mandeb. As an example, we give the results of a calculation for the Bosporus Strait under calm conditions.

To determine the 5 unknown parameters D_2, ξ_1, ξ_2, m, a, associated with the variation of the depth of the strait and of the density at its surface, we have, according to (16) and (20), 5 algebraic equations:

$$\begin{aligned} \overline{\rho}(\xi_1, 0) &= \varepsilon^2 m^{-3} \xi_1^{5a+2} D_2, \\ \overline{\rho}(\xi_2, 0) &= \varepsilon^2 m^{-3} \xi_2^{5a+2} D_2, \\ \overline{h}_1 &= m \xi_1^{-a}, \\ \overline{h}_2 &= m \xi_2^{-a}, \\ \xi_2 - \xi_1 &= 1. \end{aligned} \tag{25}$$

The coefficients of friction and diffusion were calculated by Defant (2). From observations (3), the density at the ends of the strait is known. All morphological characteristics entering into (25) are determined with the aid of nautical charts. Solving the system (25) with respect to the five indicated parameters, we find the density in the strait from the formula

$$\rho = \rho_0 e^{m^{-3} \xi^{5a+2}} f'''(\eta). \tag{26}$$

The calculated distribution of density in one of the sections of the strait is fairly close to that observed under the same conditions (Fig. 1).

In works (2,4) dealing with the study of the dynamics of currents in straits, the depth of the interface between two water masses is taken as prescribed. In the present problem it is not difficult to calculate this depth. For this it is sufficient to investigate the function $f^{IV}(\eta)$ for a maximum. According to the calculations, the change in the depth of the interface is 36.5 m/30 km; according to observations, 33 m/30 km.

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Odessa Branch of the Institute of Biology of the Southern Seas
of the Academy of Sciences of the Ukrainian SSR

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Note: Figure translations are in progress. See original paper for figures.