Soviet-era science, translated into English

V. I. Protasov

1964

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196401.64087

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

V. I. Protasov

On Solutions of Linear Equations of Infinite Order in Generalized Derivatives

(Presented by Academician A. N. Kolmogorov on 26 X 1963)

In the present article we give some theorems on the existence of solutions of a linear nonhomogeneous equation of infinite order in generalized derivatives with constant coefficients. Linear equations of infinite order in generalized derivatives of the form

$$\sum_{n=0}^{\infty} a_n D^n y(z) = f(z), \tag{1}$$

are considered, where a_n are constant coefficients. By $D^n y$ is meant the generalized derivative of order n of the function $y(z)$, which is defined as follows.

Let

$$y(z) = \sum_{k=0}^{\infty} c_k z^k$$

be a function analytic in some disk, and let a point $(\alpha_0, \alpha_1, \alpha_2, ...)$, $\alpha_k \neq 0$, $k = 0, 1, 2, ...$, be given, where α_k are complex numbers. Then

$$D^n y(z) = \sum_{k=n}^{\infty} \frac{\alpha_{k-n}}{\alpha_k} c_k z^{k-n},$$

provided that the series

$$\sum_{k=n}^{\infty} \frac{\alpha_{k-n}}{\alpha_k} c_k z^{k-n}$$

converges in some disk.

Equations of the form (1) were studied in work (1). In the present article we give results obtained under other restrictions and by other methods.

We shall assume that the right-hand side of equation (1)—the function $f(z)$—is analytic in some disk or entire. The solutions $y(z)$ will be considered among

SovietRxiv

analytic functions, and the series on the left-hand side of the equation will be regarded as absolutely convergent in some disk. Let

$$f(z) = \sum_{j=0}^{\infty} b_j z^j, \qquad y(z) = \sum_{k=0}^{\infty} c_k z^k$$

be analytic functions. Associate with the function $f(z)$ the point $(b_0, b_1, b_2, \dots, b_j, \dots)$, and with the function $y(z)$ the point $(c_0, c_1, c_2, \dots, c_k, \dots)$. Then equation (1) may be considered as a linear transformation of the point $(c_0, c_1, c_2, \dots)$ into the point $(b_0, b_1, b_2, \dots)$ by an infinite matrix represented by the product of three matrices, and written in the form

$$\begin{bmatrix} \alpha_0 & & & & \\ & \alpha_1 & & & \\ & & \alpha_2 & & \\ & & & \alpha_3 & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \dots & \\ & a_0 & a_1 & a_2 \dots & \\ & & a_0 & a_1 \dots & \\ & & & a_0 \dots & \\ & & & & \dots \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha_0} & & & & \\ & \frac{1}{\alpha_1} & & & \\ & & \frac{1}{\alpha_2} & & \\ & & & \frac{1}{\alpha_3} & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}.$$

The same transformation can be written in the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \dots & \\ & a_0 & a_1 & a_2 \dots & \\ & & a_0 & a_1 \dots & \\ & & & a_0 \dots & \\ & & & & \dots \end{bmatrix} \begin{bmatrix} \frac{c_0}{\alpha_0} \\ \frac{c_1}{\alpha_1} \\ \frac{c_2}{\alpha_2} \\ \frac{c_3}{\alpha_3} \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{b_0}{\alpha_0} \\ \frac{b_1}{\alpha_1} \\ \frac{b_2}{\alpha_2} \\ \frac{b_3}{\alpha_3} \\ \vdots \end{bmatrix}. \qquad (2)$$

The linear transformation (2) of the point $(c_0, c_1, c_2, \dots, c_k, \dots)$ into the point $(b_0, b_1, b_2, \dots, b_j, \dots)$ is studied as a transformation of perfect sequence spaces considered in ([2,3]).

The theorems obtained by the author of this paper on transformations of perfect spaces, as applied to transformation (2), make it possible to obtain the following theorems on the existence of solutions of equation (1).

By $\varphi(t)$ we shall denote the characteristic function of the equation, assuming

SovietRxiv

$$\varphi(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Theorem 1. If

$$\underline{\lim}_{n\to\infty} n^{1/\rho} \sqrt[n]{|\alpha_n|} = (\sigma e \rho)^{1/\rho} > 0, \qquad \overline{\lim}_{n\to\infty} n^{1/\rho} \sqrt[n]{|\alpha_n|} = (\bar{\sigma} e \rho)^{1/\rho},$$

and the characteristic function $\varphi(t)$ of equation (1) is of integral order of growth ρ and type σ, then for the right-hand side $f(z)$, analytic in the disk $|z| < R$, where

$$R > \left(\frac{2\sigma}{\sigma}\right)^{1/\rho},$$

there exists a solution analytic at least in the disk

$$|z| < \left(R^\rho \frac{\sigma}{\sigma} - \frac{\sigma}{\sigma}\right)^{1/\rho}.$$

Let us give the simplest example, which confirms the sharpness of the theorem. Let $\varphi(t) = e^t$ $(\rho = 1,\ \sigma = 1)$, $\alpha_n = \dfrac{1}{n!}$. The generalized derivative in this case reduces to the ordinary one. The solution of the differential equation

$$\sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)} = f(z)$$

can be written in the form $y(z) = f(z-1)$. If the function $f(z)$ is analytic only in the disk $|z| < R$, where $R > 2$, then the solution is analytic in the disk $|z| < R - 1$, and the radius $R - 1$ cannot be increased.

It follows from Theorem 1 that if the right-hand side of equation (1) is an entire function, then there exists a solution among entire functions. Under other assumptions concerning the growth of the moduli of the numbers α_n, one can obtain the following theorems.

Theorem 2. If

$$\underline{\lim}_{n\to\infty} n^{1/\rho_1} \sqrt[n]{|\alpha_n|} = \delta > 0, \qquad \overline{\lim}_{n\to\infty} n^{1/\rho_2} \sqrt[n]{|\alpha_n|} = \bar{\delta} > 0,$$

$$\overline{\lim}_{n\to\infty} n^{1/\rho} \sqrt[n]{|a_n|} = (\sigma e \rho)^{1/\rho}, \qquad \text{where } \rho_1 \leqslant \rho_2 < \rho,$$

SovietRxiv

then equation (1), with right-hand side $f(z)$ entire of order of growth $\frac{\rho\rho_1}{\rho-\rho_1}$ and type σ_0,

$$\sigma_0 < \frac{\rho-\rho_1}{e\rho\rho_1}\left[\frac{\delta}{(2\sigma e\rho)^{1/\rho}}\right]^{\rho\rho_1/(\rho-\rho_1)},$$

has a solution among entire functions of order of growth not exceeding $\frac{\rho\rho_2}{\rho-\rho_2}$ and of normal type.

Theorem 3. If

$$\underline{\lim}_{n\to\infty} n^{1/\rho_1}\sqrt[n]{|\alpha_n|} = \delta > 0; \qquad \overline{\lim}_{n\to\infty} n^{1/\rho}\sqrt[n]{|\alpha_n|} = \bar{\delta} > 0, \quad \rho > \rho_1;$$

$$\overline{\lim}_{n\to\infty} n^{1/\rho}\sqrt[n]{|\alpha_n|} = (\sigma e\rho)^{1/\rho},$$

then equation (1), with right-hand side $f(z)$ entire of order of growth $\frac{\rho\rho_1}{\rho-\rho_1}$ and type σ_0

$$\sigma_0 < \frac{\rho-\rho_1}{e\rho\rho_1}\left[\frac{\delta}{(2\sigma e\rho)^{1/\rho}}\right]^{\rho\rho_1/(\rho-\rho_1)},$$

has a solution analytic in the disk

$$|z| < \frac{1}{\bar{\bar{\delta}}}\left\{\left[\left(\frac{\delta}{B}\right)^{\rho}\frac{1}{e\rho} - \sigma\right] e\rho\right\}^{1/\rho}, \qquad \text{where } B = \left(\sigma_0 e\frac{\rho\rho_1}{\rho-\rho_1}\right)^{(\rho-\rho_1)/\rho\rho_1}.$$

Theorem 4. If

$$\overline{\lim}_{n\to\infty} n^{1/\rho}\sqrt[n]{|\alpha_n|} = (\sigma e\rho)^{1/\rho}, \qquad \underline{\lim}_{n\to\infty} n^{1/\rho}\sqrt[n]{|\alpha_n|} = (\underline{\sigma} e\rho)^{1/\rho},$$

$$\overline{\lim}_{n\to\infty} n^{1/\rho}\sqrt[n]{|\alpha_n|} = (\bar{\sigma} e\rho)^{1/\rho}$$

and the function $f(z)$ has growth $[\rho_0, \sigma_0]$, then there exists a solution of equation (1) among entire functions of growth not exceeding order ρ_0 and of normal type; moreover, for $\rho_0 < \rho$ the solution is unique.

For $\rho = \rho_0$ the solution is unique if there exists such an r, $r > \left(\frac{\sigma_0}{\underline{\sigma}}\right)^{1/\rho}$, that in the disk $|t| < r$ the characteristic function $\varphi(t)$ has no zeros.

Rostov Higher Command Engineering School

Received
1 VII 1963

REFERENCES

1. Yu. N. Frolov, *Vestn. Mosk. univ.*, No. 4, 3 (1960).

2. G. Köthe, O. Toeplitz, *J. reine u. angew. Math.*, 171, 193 (1934).

3. R. Cook, *Infinite Matrices and Sequence Spaces*, Moscow, 1960.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.