ON ANGULAR LIMIT VALUES OF INTEGRALS OF CAUCHY TYPE

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Abstract

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ON ANGULAR LIMIT VALUES OF INTEGRALS OF CAUCHY TYPE

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In the present note some necessary and sufficient conditions are indicated for the existence of angular limit values of integrals of Cauchy type at a prescribed point of a curve, under the assumption that the corresponding densities of the integrals belong to certain classes of functions on the curve. For simplicity we shall confine ourselves to the case where C is a simple rectifiable oriented arc in the Euclidean plane E_2, which we identify with the set of complex numbers. For $M \subset C$, denote by λM the length (linear measure) of the set M (see ([5])). We agree that φ will always denote some finite (in general complex-valued) function, integrable (λ) on C. If $\eta \in C$, then, by definition, we put

$$\Phi_\eta(z) = \int_C \frac{\varphi(\xi) - \varphi(\eta)}{\xi - z}\, d\xi, \qquad z \in E_2 \setminus C.$$

If it is known that φ assumes on C only real values, then we also put

$$\Phi^1_\eta(z) = \operatorname{Re} \Phi_\eta(z), \qquad \Phi^2_\eta(z) = \operatorname{Im} \Phi_\eta(z).$$

Let p be a bounded, nonnegative, lower semicontinuous function defined on C. We shall investigate the behavior of the integrals $\Phi_\eta(z)$, $\Phi^k_\eta(z)$ as z approaches η along nontangential paths, under the assumption that φ satisfies one of the conditions

$$\varphi(\xi) - \varphi(\eta) = O(p(\xi)), \qquad \xi \to \eta; \tag{1}$$

$$\varphi(\xi) - \varphi(\eta) = o(p(\xi)), \qquad \xi \to \eta. \tag{2}$$

For this we shall need the quantities $v^k_R(\eta; p)$, $V^k_R(\eta; p)$, to whose definition we now proceed.

If $a \le b$ and $R > 0$, then we put

$$S_R(\eta; a, b) = \{\eta + re^{iy};\ 0 < r < R,\ a \le y \le b\}.$$

For real x and $R > 0$, denote by $\Gamma_R(x;\eta,p)$ the sum

$$\sum_\xi p(\xi), \qquad \xi \in C \cap S_R(\eta;x,x)$$

(of course, $\Gamma_R(x;\eta,p) = +\infty$ if $p(\xi) > 0$ for an uncountable set of points $\xi \in C \cap S_R(\eta;x,x)$). Analogously we put

$$\gamma_R(x;\eta,p) = \sum_\xi |\xi - \eta|\, p(\xi), \qquad \xi \in C \cap S_R(\eta;x,x).$$

For each $x > 0$ we also put

$$v(x;\eta,p) = \sum_\xi p(\xi), \qquad \xi \in C, \quad |\xi - \eta| = x.$$

The functions $\Gamma_R(x;\ldots)$, $\gamma_R(x;\ldots)$, $v(x;\ldots)$ are Lebesgue measurable with respect to x. Put

$$v_R(\eta;p) = \int_0^{2\pi} \gamma_R(x;\eta,p)\,dx, \qquad v_R^2(\eta;p) = \int_0^R v(x;\eta,p)\,dx,$$

$$V_R^1(\eta;p) = \int_0^R x^{-1} v(x;\eta,p)\,dx, \qquad V_R^2(\eta;p) = \int_0^{2\pi} \Gamma_R(x;\eta,p)\,dx.$$

Obviously, the quantities $v_R^k(\ldots)$, $V_R^k(\ldots)$ do not decrease as R increases. We are now in a position to formulate the following theorems, in which we always assume that $\eta \in C$, $R > 0$, $a < b$, and C does not intersect $S_R(\eta;a,b)$. In Theorems 1 and 3 we also assume that C does not intersect $S_R(\eta;a+\pi,b+\pi)$.

Theorem 1. Let $a < x < b$, $k > 0$. If, for every real function φ satisfying the inequality

$$|\varphi(\xi) - \varphi(\eta)| \le K p(\xi), \qquad \xi \in C, \tag{3}$$

the relation

$$\limsup_{r \to 0+} \left|\Phi_\eta^k(\eta + re^{ix})\right| < \infty$$

holds, then necessarily

$$V_\infty^k(\eta;p) < \infty, \tag{4_k}$$

$$\sup_{r>0} r^{-1} v_r^k(\eta;p) < \infty. \tag{5}$$

The following converse of this theorem is also valid:

Theorem 2. If (5) holds and, for some $\rho > 0$,

$$V_\rho^k(\eta;p) < \infty, \tag{6_k}$$

then, for every real function φ satisfying condition (1), the corresponding function Φ_η^k is bounded on $S_r(\eta;a_1,b_1)$, where $a < a_1 \leq b_1 < b$, $0 < r < R$; all functions Φ_η^k corresponding to functions φ satisfying inequality (3) (with one and the same constant K) are uniformly bounded on $S_r(\eta;a_1,b_1)$. If, instead of (1), one requires (2), then $\Phi_\eta^k(z)$ extends continuously to the point η along the set $S_r(\eta;a_1,b_1)$. If (5) is replaced by the condition

$$v_\rho^k(\eta;p) = o(\rho), \qquad \rho \to 0+,$$

then all functions Φ_η^k corresponding to functions φ satisfying (3) are equicontinuous on $S_r(\eta;a_1,b_1)$.

From these theorems the following propositions follow:

Theorem 3. Let $a < x < b$. If

$$\limsup_{r\to 0+} \left|\Phi_\eta(\eta + re^{ix})\right| < \infty$$

for every function φ satisfying (3), then (4_1) and (4_2) hold.

Theorem 4. If, for some $\rho > 0$, (6_1) and (6_2) hold, then, for every function φ satisfying condition (1), the relation

$$\int_C \frac{\varphi(\xi) - \varphi(\eta)}{\xi - \eta}\, d\xi = \lim_{z\to\eta} \Phi_\eta(z), \qquad z \in S_R(\eta;a_1,b_1),$$

holds, where $a < a_1 \leq b_1 < b$. All functions Φ_η corresponding to functions φ satisfying (3) (with one and the same constant K) are equicontinuous on $S_r(\eta;a_1,b_1)$, where $0 < r < R$.

If the function p has the form $p(\zeta) = h(|\zeta - \eta|)$, where h is a function of one real variable, then conditions (4_1), (4_2) in Theorem 3 and conditions (6_1), (6_2) in Theorem 4 can be replaced by the single condition

$$\int_0^\infty \rho^{-1} h(\rho)\, dm(\rho) < \infty,$$

where $m(\rho) = \lambda\{\zeta;\ \zeta \in C,\ |\zeta - \eta| < \rho\}$.

The proofs of these theorems use methods from the theory of functions of a real variable connected with Banach' s theorem on the total variation of a continuous function.

The quantities $V_R^2(\eta; p)$, $v_R^2(\eta; p)$ for the particular case $p \equiv 1$ were introduced in ([2]). The quantity corresponding to $V_\infty^2(\eta; 1)$ in three-dimensional space was considered in ([1]). For references to studies on integrals of Cauchy type, see the monographs ([3,4]) and the survey article ([6]).

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Note: Figure translations are in progress. See original paper for figures.