Reports of the Academy of Sciences of the USSR

MATHEMATICS

1964

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1964. Volume 155, No. 2

MATHEMATICS

B. I. MORGUNOV

STATIONARY RESONANT REGIMES OF CERTAIN ROTATIONAL MOTIONS

(Presented by Academician N. N. Bogolyubov, November 23, 1963)

§ 1. **Statement of the problem.** In papers ($^{1-9}$) the asymptotics of certain rotational motions with one degree of freedom, depending on slowly varying parameters, was considered. In ($^{4-6}$) equations were obtained which describe, in the first and second approximations (with an error of order ε and ε^2, respectively, where $\varepsilon > 0$ is a small parameter), the slow variation of the energy of the perturbed motion; the phase and period of the perturbed motion were also found, as well as the first approximations for the coordinate and velocity. However, in ($^{4-6}$) the cases in which the perturbation depends explicitly on time were not considered. In the present paper, based on the methods developed in (^{7}), the case is investigated in which the perturbation depends periodically on time, and the question of the existence and stability (in a certain generalized sense) of stationary resonant regimes of such systems is considered.

Let the rotational motion be described by a system of the form

$$\frac{d}{dt}\left[m(x)\dot{y}\right] + Q(x, y) = \varepsilon f(x, y, \dot{y}, \vartheta), \qquad \dot{x} = \varepsilon X(x, y, \dot{y}, \vartheta),$$

$$\dot{\vartheta} = \nu(x) + \varepsilon\Theta(x, y, \dot{y}, \vartheta). \tag{1}$$

Here y is a one-dimensional coordinate; $m(x)$ is the mass; $x = (x_1, \ldots, x_n)$ is a collection of slowly varying parameters; $Q(x, y)$ is the potential force causing the rotation; f is a small nonlinear perturbation; ϑ is the phase of the external perturbing force. All functions entering into (1) are assumed to be periodic in y and ϑ with period 2π, and Q has zero mean value in y.

§ 2. **Main results.** With the aid of a certain change of variables (see, for example, ($^{4-6}$)), system (1) can be rewritten in the form

$$\dot{E} = \varepsilon G(x, y, E, \vartheta), \qquad \dot{x} = \varepsilon X\left(x, y, \sqrt{\frac{2}{m}(E - V)}, \vartheta\right), \tag{2}$$

$$\dot{\psi} = \omega(E,x) + \varepsilon\Psi(E,x,y,\vartheta), \qquad \dot{\vartheta} = \nu(x) + \varepsilon\Theta\left(x, y, \sqrt{\frac{2}{m}(E-V)}, \vartheta\right),$$

where

$$\frac{\partial V}{\partial y} \equiv Q; \qquad T = \int_0^{2\pi} \frac{dy}{\sqrt{\frac{2}{m}(E-V)}} \text{ is the period of rotation;} \qquad \omega = \frac{2\pi}{T};$$

y is determined by integrals of the unperturbed system; E is the energy; ψ is the phase of rotation;

$$G = \frac{1}{m}\left(-E\frac{\partial m}{\partial x} + \frac{\partial mV}{\partial x}\right) X\left(x, y, \sqrt{\frac{2}{m}(E-V)}, \vartheta\right) + \sqrt{\frac{2}{m}(E-V)}\, f\left(x, y, \sqrt{\frac{2}{m}(E-V)}, \vartheta\right),$$

$$\Psi = \frac{2\pi}{T}\left[\int_{y_0}^{y} \frac{\frac{E}{m^2}\frac{\partial m}{\partial x} + \frac{\partial}{\partial x}\left(\frac{V}{m}\right)}{\left(\frac{2}{m}(E-V)\right)^{3/2}}\, d\eta - \frac{1}{T}\int_0^{2\pi} \frac{\frac{E}{m^2}\frac{\partial m}{\partial x} + \frac{\partial}{\partial x}\left(\frac{V}{m}\right)}{\left(\frac{2}{m}(E-V)\right)^{3/2}}\, dy \int_{y_0}^{y} \frac{d\eta}{\sqrt{\frac{2}{m}(E-V)}}\right] \times$$

$$\times X\left(x, y, \sqrt{\frac{2}{m}(E-V)}, \vartheta\right) +$$

$$+\frac{2\pi}{mT}\left[\frac{1}{T}\int_0^{2\pi} \frac{dy}{\left(\frac{2}{m}(E-V)\right)^{3/2}} \int_{y_0}^{y} \frac{d\eta}{\sqrt{\frac{2}{m}(E-V)}} - \int_{y_0}^{y} \frac{d\eta}{\left(\frac{2}{m}(E-V)\right)^{3/2}}\right] G(E,x,y,\vartheta).$$

We shall say that a resonance characterized by relatively prime integers p and q occurs in the system if, for certain values of the parameters E, x, the equality $p\omega = q\nu$ is satisfied. Passing from the variables ψ, ϑ to the variables $\varphi = \vartheta - \frac{p}{q}\psi$, $\beta = \frac{1}{q}\psi$, we rewrite (2) in the form

$$\dot{E} = \varepsilon G(E,x,y,\varphi + p\beta), \qquad \dot{x} = \varepsilon X\left(x, y, \sqrt{\frac{2}{m}(E-V)}, \varphi + p\beta\right),$$

$$\dot{\varphi} = \lambda(E,x) + \varepsilon\Phi(E,x,y,\varphi + p\beta), \qquad \dot{\beta} = \Omega(E,x) + \varepsilon B(E,x,y,\varphi + p\beta), \quad (3)$$

where $\lambda = \nu - \frac{p}{q}\omega$, $\Omega = \frac{1}{q}\omega$, $\Phi = \Theta - \frac{p}{q}\Psi$, $B = \frac{1}{q}\Psi$.

We introduce, in the following way, the mean values of the functions appearing on the right-hand sides of (3), for fixed values of the parameters E_0, x_0:

$$G_1(E_0, x_0, \varphi) = \tag{4}$$

$$= \frac{1}{qT(E_0, x_0)} \int_{y_0}^{y_0+2\pi q} G\left(E_0, x_0, y, \varphi + p\beta_0 + \nu_0 \int_{y_0}^{y} \frac{d\eta}{\sqrt{\frac{2}{m}(E_0 - V)}}\right) \frac{dy}{\sqrt{\frac{2}{m}(E_0 - V)}},$$

where $\beta_0 = \beta|_{t=t_0}$. The mean values of the functions X, Φ and of the derivatives $\partial G/\partial x_0$, $\partial G/\partial E_0$, etc., are defined analogously.

We apply to system (3) the scheme developed in (7) for systems of a more general form. The resonant values of the energy E_0, of the parameters x_0, and of the detuning φ_0 in the zeroth approximation in ε are found from the system

$$G_1(E_0, x_0, \varphi_0) = 0, \qquad X_1(E_0, x_0, \varphi_0) = 0, \qquad \lambda(E_0, x_0) = 0. \tag{5}$$

The corrections to the coordinates of the resonant point $\delta E, \delta x, \delta\varphi$ in the first approximation are found from the linear system of equations

$$\frac{\partial G_1}{\partial E_0}\delta E + \frac{\partial G_1}{\partial x_0}\delta x + \frac{\partial G_1}{\partial \varphi_0}\delta\varphi = 0, \qquad \frac{\partial X_1}{\partial E_0}\delta E + \frac{\partial X_1}{\partial x_0}\delta x + \frac{\partial X_1}{\partial \varphi_0}\delta\varphi = 0,$$

$$\frac{\partial \lambda}{\partial E_0}\delta E + \frac{\partial \lambda}{\partial x_0}\delta x + \Phi_1(E_0, x_0, \varphi_0) = 0. \tag{6}$$

We introduce, in accordance with (7), the characteristic equation

$$\det(A - kE) = 0, \tag{7}$$

where E is the identity matrix of the corresponding dimension, and A is a matrix of the form

$$A = \begin{pmatrix} \varepsilon\dfrac{\partial G_1}{\partial E_0} & \varepsilon\dfrac{\partial G_1}{\partial x_0} & \varepsilon\dfrac{\partial G_1}{\partial \varphi_0} \\ \varepsilon\dfrac{\partial X_1}{\partial E_0} & \varepsilon\dfrac{\partial X_1}{\partial x_0} & \varepsilon\dfrac{\partial X_1}{\partial \varphi_0} \\ \dfrac{\partial \lambda}{\partial E_0} + \varepsilon\left(\dfrac{\partial \Phi_1}{\partial E_0} + \dfrac{\partial^2 \lambda}{\partial E_0 \partial x_0}\delta x + \dfrac{\partial^2 \lambda}{\partial E_0^2}\delta E\right) & \dfrac{\partial \lambda}{\partial x_0} + \varepsilon\left(\dfrac{\partial \Phi_1}{\partial x_0} + \dfrac{\partial^2 \lambda}{\partial x_0^2}\delta x + \dfrac{\partial^2 \lambda}{\partial x_0 \partial E_0}\delta E\right) & \varepsilon\dfrac{\partial \Phi_1}{\partial \varphi_0} \end{pmatrix}.$$

We require that systems (5) and (6) be solvable and that all roots (7) have negative real parts:

$$\operatorname{Re} k < 0. \tag{8}$$

We shall call condition (8) the stability condition for the stationary regime (3). Here stability is understood in the following sense: from results (7) it follows that for arbitrarily large $T > 0$ and arbitrarily small $\xi > 0$ there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ there exists a $\delta(\varepsilon)$ such that from the condition

$$\max |E(t_0) - E_0,\ x(t_0) - x_0,\ \varphi(t_0) - \varphi_0| < \delta$$

for all $t_0 \leq t \leq T$ there follows the inequality

$$\max |E(t) - E_0,\ x(t) - x_0,\ \varphi(t) - \varphi_0| < \xi.$$

The statement just given can be refined: if the initial data for (3) are given in some ε-neighborhood of the resonant point E_0, x_0, φ_0, then on a time interval of order $\varepsilon^{-1/2}$ the solutions of (3) will not leave some neighborhood of the resonant point of size of order $\varepsilon^{1/2}$.

§ 3. **The case of slow time.** Let the only slowly varying parameter be the slow time $\tau = \varepsilon t$. Then the coordinates of stationary resonant points are found from the equations

$$\frac{\partial\lambda/\partial\tau}{\partial\lambda/\partial E_0} + G_1(E_0, \varphi_0, \tau) = 0, \qquad \lambda(E_0, \tau) = 0,$$

where in this case $E_0 = E_0(\tau), \varphi_0 = \varphi_0(\tau)$. The conditions ensuring (8) take the form:

$$\left(\frac{\partial G_1}{\partial E_0} + \frac{\partial \Phi_1}{\partial \varphi_0}\right)_{\tau=\tau_0} < 0, \qquad \left(\frac{\partial\lambda}{\partial E_0}\frac{\partial G_1}{\partial \varphi_0}\right)_{\tau=\tau_0} < 0.$$

The principal assertion made in § 2 is also valid in this case.

§ 4. **The case of large energies.** Let us consider the case of large energies, important for applications, using as an example a perturbation of the special form:

$$\ddot{y} + Q(y) = \varepsilon[f(\nu t) - \lambda\dot{y}]. \tag{9}$$

Here Q satisfies the conditions listed in § 1, f is periodic in νt with period 2π, and $\lambda > 0$. The asymptotics at large energies for (9) was studied by other methods by N. N. Moiseev in (^{1}), where stationary regimes were investigated under the assumption that the parameter values differ from the resonant values by quantities of order ε.

Applying the results of § 2 to (9), we obtain the equations for determining the coordinates of the stationary point in the zero approximation in the form

$$-\frac{p}{q}\sqrt{2E_0}+\nu+\frac{1}{\sqrt{2E_0}}\frac{p\bar{V}}{q}=0, \qquad \bar{V}=\frac{1}{2\pi}\int_0^{2\pi}V(y)\,dy,$$

$$-2\lambda E_0+\sqrt{2E_0}f(\varphi_0)+2\lambda\bar{V}+\pi\nu q f'(\varphi_0)+\frac{1}{\sqrt{2E_0}}\left(\frac{2\pi^2\nu^2q^2f''(\varphi_0)}{3}-\bar{V}f(\varphi_0)\right)=0. \tag{10}$$

(Here and below it is assumed that $y|_{t=t_0}=O(\varepsilon)$, $\varphi=\vartheta-\frac{p}{q}\psi+p\beta_0$.)

The stability condition for the stationary resonant regime (10) has the form:

$$f'(\varphi_0)+\frac{\pi\nu q}{\sqrt{2E_0}}f''(\varphi_0)>0.$$

§ 5. **Examples.** As a first example, consider the mathematical pendulum in a rotational regime, studied by another method in (^{2}). The equation of motion has the form

$$\ddot{y}+\sin y=\varepsilon(\sin\nu t-\lambda\dot{y}).$$

The coordinates of stationary resonant points are found from the equations ($q=1$)

$$\sqrt{2E_0}=\frac{2\nu}{\pi p}K\left(\sqrt{\frac{2}{E_0}}\right),$$

$$\int_0^{2\pi}\sin\left(\varphi_0+\nu\int_0^y\frac{d\eta}{\sqrt{2(E_0-V)}}\right)dy=4\lambda\sqrt{2E_0}\,G\left(\sqrt{\frac{2}{E_0}}\right).$$

Here K is the complete elliptic integral of the first kind, G is the complete elliptic integral of the second kind. Let us write out the stability conditions for these regimes:

$$\frac{\partial T}{\partial E_0}\int_0^{2\pi}\cos(\varphi_0+\nu I_0)\,dy<0,$$

$$\lambda-\frac{\nu}{T_0}\int_0^{2\pi}\cos(\varphi_0+\nu I_0)\frac{\partial I}{\partial E_0}\,dy+\frac{2\pi p}{T_0}\int_0^{2\pi}\cos(\varphi_0+\nu I_0)\frac{\partial T^{-1}I}{\partial E_0}\,dy>0,$$

where

$$I = \sqrt{\frac{2}{E}}\, F\left(\frac{y}{2}, \sqrt{\frac{2}{E}}\right), \qquad T = \frac{4}{\sqrt{2E}} K\left(\sqrt{\frac{2}{E}}\right),$$

F is an elliptic integral of the first kind. The case of large energies for the mathematical pendulum is easily investigated with the aid of the formulas of § 3.

As a second example, consider the planar motion of a satellite relative to the center of inertia, moving in a central gravitational field along an orbit close to circular, investigated in (^{3}). The equation of motion of the satellite has the form

$$\ddot{y} + 3a^2 \sin y = \varepsilon(4 \sin t + 2 \sin t\, \dot{y} + 3a^2 \cos t \sin y),$$

where $\varepsilon \ll 1$ is the small eccentricity of the orbit, t is the angular distance of the radius vector of the center of mass from the perigee of the orbit, y is twice the angle between the radius vector of the center of mass and one of the axes of inertia, $a = \text{const}$. The stationary resonant regimes are found from the equations $(q = 1)$

$$\sqrt{2E_0} = \frac{2}{\pi p} K\left(\sqrt{\frac{6a^2}{E_0}}\right), \tag{11}$$

$$\int_0^{2\pi} \left[2\left(\sqrt{2(E_0 - V)} + 2\right)\sin(\varphi_0 + I_0) + 3a^2 \sin y \cos(\varphi_0 + I_0)\right] dy = 0.$$

The stability conditions of the stationary regimes take the form

$$\frac{\partial T}{\partial E_0}\int_0^{2\pi} \left[2\left(\sqrt{2(E_0 - V)} + 2\right)\cos(\varphi_0 + I_0) - 3a^2 \sin y \sin(\varphi_0 + I_0)\right] dy < 0,$$

$$\begin{aligned} \int_0^{2\pi} \Bigg[&\left(\frac{2}{\sqrt{2(E_0 - V)}} - 3a^2 \sin y\, \frac{\partial I}{\partial E_0}\right)\sin(\varphi_0 + I_0) \\ &+ 2\left(\sqrt{2(E_0 - V)} + 2\right)\frac{\partial I}{\partial E_0}\cos(\varphi_0 + I_0)\Bigg] dy \\ - 2\pi p \int_0^{2\pi} &\frac{\partial T^{-1} I}{\partial E_0}\left[2\left(\sqrt{2(E_0 - V)} + 2\right)\cos(\varphi_0 + I_0)\right. \\ &\left. -3a^2 \sin y \sin(\varphi_0 + I_0)\right] dy < 0; \end{aligned} \tag{12}$$

$$T = \frac{4}{\sqrt{2E}} K\left(\sqrt{\frac{6a^2}{E}}\right), \qquad I = \sqrt{\frac{2}{E}}\, F\left(\frac{y}{2}, \sqrt{\frac{6a^2}{E}}\right).$$

(In (11) and (12) the notation of the first example is preserved.)

I take this opportunity to express my deep gratitude to V. M. Volosov for posing the problem and discussing the results.

Moscow State University
named after M. V. Lomonosov

Received
14 XI 1963

References

[1] N. N. Moiseev, *Zhurnal vychislitel' noi matematiki i matematicheskoi fiziki*, **3**, No. 1 (1963).
[2] F. L. Chernous' ko, *Zhurnal vychislitel' noi matematiki i matematicheskoi fiziki*, **3**, No. 1 (1963).
[3] F. L. Chernous' ko, *Zhurnal vychislitel' noi matematiki i matematicheskoi fiziki*, **3**, No. 3 (1963).
[4] V. M. Volosov, B. I. Morgunov, *Doklady Akademii Nauk SSSR*, **151**, No. 6 (1963).
[5] B. I. Morgunov, *Vestnik MGU*, No. 6 (1963).
[6] B. I. Morgunov, *Vestnik MGU*, No. 1 (1964).
[7] V. M. Volosov, B. I. Morgunov, *Doklady Akademii Nauk SSSR*, **153**, No. 3 (1963).
[8] V. M. Volosov, *Uspekhi matematicheskikh nauk*, **17**, issue 6 (108) (1962).
[9] V. M. Volosov, *Zhurnal vychislitel' noi matematiki i matematicheskoi fiziki*, **3**, No. 1 (1963).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.