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Abstract

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ALGORITHMIC UNDECIDABILITY OF THE PROBLEM OF RECOGNIZING COMPLETENESS FOR FINITE AUTOMATA

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1. The concepts of a "logical network over a basis consisting of delay elements and logical elements" and of "realization of an operator in a logical network" were first formulated by Burks and Wright ([1]). In the present article we shall use these terms in a more general sense, which will be clarified below ([2]).

First, the elements may be arbitrary finite automata, not necessarily logical elements and delay elements, as in ([1]). Let us recall that, in forming feedbacks in logical networks, the condition of correctness of cycles is very essential. In ([1]) this is expressed in the requirement that every directed cycle of the logical network contain at least one delay element. In accordance with ([2]), in our considerations this will be expressed in the requirement that every directed cycle of the logical network pass through an element whose output channel is not subordinated to that input channel of it which lies on this cycle (see ([2]), p. 122).

Second, each input and each output channel of an element may be in states from some finite set of states M (the finite alphabet M), not necessarily the alphabet $\{0, 1\}$, as in ([1]). To emphasize this circumstance we shall use the terms M-element, M-basis, M-operator. Let us note that if in ([1]) the operators realized in logical networks transform sequences $x(1), x(2), ...$ into sequences $z(1), z(2), ...$, where $x(t), z(t)$ are tuples of zeros and ones, then in the case of an M-operator $x(t), z(t)$ are tuples of letters of the alphabet M. In what follows, where one speaks simply of bases, elements, operators, it is understood that M is an arbitrary fixed alphabet.

Third, with respect to bases one may adopt either of the following two conventions: 1) the elements of the basis are initial automata, and the initial state of a logical network over the basis is the state in which all its elements are in their initial states (initial basis); 2) the elements of the basis are not initial automata and may enter a logical network having as their initial state any one of their internal states (noninitial basis).

If there exists such a logical network over an initial (noninitial) basis $\mathfrak{B}$ in which an operator T is realized, then we shall say that T is **realizable in the initial (noninitial) basis $\mathfrak{B}$**. An initial (noninitial) basis is called **complete** if every o-d operator is realizable in it*. In the theory of finite automata an important place is occupied by the completeness problem, which consists in finding conditions for completeness of bases.

More particular formulations of the completeness problem are also possible, as applied to special classes of operators and to stricter restrictions on the rules for forming logical networks.

At present the completeness problem has been solved precisely in such particular formulations. Completeness criteria for functions of the algebra of logic were established—

* Let us recall that if an M-basis is in question, then automaton M-operators are realized in it.

by E. Post ([3]) and S. V. Yablonskii ([4]). Since the functions of the algebra of logic are adequate to automata without memory, the completeness problem for this class of automata turned out to be solved. This circumstance was noted by S. V. Yablonskii ([4]). Important work toward establishing completeness criteria for arbitrary sets of objects was carried out by A. V. Kuznetsov*. V. B. Kudryavtsev, using the results of A. V. Kuznetsov and S. V. Yablonskii, established a completeness criterion for one particular type of automata—automata without feedback ([5]). A. A. Letichevskii formulated completeness conditions for bases containing Moore automata and automata without memory ([6]).

2. With the search for completeness criteria one can naturally associate the following mass problem (the problem of recognizing completeness): it is required to find an algorithm which, for any given basis, makes it possible to determine whether it is complete or not**. Of course, under various particular refinements of the concept of completeness this problem must be interpreted in the appropriate way.

It should be noted that in the works cited above ([3–6]) the completeness criteria found proved in all cases to be effective, i.e., algorithms for solving the corresponding problem of recognizing completeness are extracted from them. The purpose of the present note is to show that, in the general case, the problem of recognizing completeness is algorithmically unsolvable. At the same time, the algorithmic unsolvability of certain other problems connected with questions of the structural synthesis of automata and with the completeness problem is established.

With every o-d operator T we associate the so-called **problem of recognizing the realization of the operator T in initial (noninitial) bases**: it is required to find an algorithm which, for any given initial (noninitial) basis $\mathfrak{B}$, makes it possible to determine whether the given o-d operator T is realizable in the basis $\mathfrak{B}$ or not (see the footnote on p. 35).

The following main theorem holds:

Theorem 1. *For an arbitrary o-d operator T, the problem of recognizing its realization in initial (noninitial) bases is algorithmically unsolvable.*

Hence it follows:

Theorem 2. *The problem of recognizing completeness for initial (noninitial) bases is algorithmically unsolvable.*

By $\mathfrak{M}_{\mathfrak{B}}$ we shall denote the set of all o-d operators that can be realized in the given basis $\mathfrak{B}$.

Theorem 3. *Each of the following three problems is algorithmically unsolvable as applied to arbitrary initial (noninitial) bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$:*

1) *Does $\mathfrak{M}_{\mathfrak{B}1} = \mathfrak{M}_{\mathfrak{B}2}$ hold or not.*
2) *Does $\mathfrak{M}_{\mathfrak{B}1} \subset \mathfrak{M}_{\mathfrak{B}2}$ hold or not.*
3) *Does $\mathfrak{M}_{\mathfrak{B}1} \cap \mathfrak{M}_{\mathfrak{B}2} = \emptyset$ hold or not.*

3. The proof of Theorem 1 is based on the fact that another algorithmically unsolvable problem—the Post correspondence problem—can be reduced to the problem of recognizing realization; it consists in the following. Let $S = P_1, Q_1; P_2, Q_2, \ldots; P_n, Q_n$ be a system of pairs of finite words in some finite alphabet M. The system S is called **correspondable** if there exists a positive integer t and such $r_1, r_2, \ldots, r_t$, taking values from the sequence $1, 2, \ldots, n$, that

$$P_{r_1} P_{r_2} \ldots P_{r_t} = Q_{r_1} Q_{r_2} \ldots Q_{r_t}.$$

The Post correspondence problem is the problem of recognizing the correspondability of arbitrary systems of pairs of words in a given alphabet M. It is known that this problem is algorithmically unsolvable if the alphabet M contains more than one letter ([7]).

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* See *Report on the Symposium on General Algebra*, UMN, **16**, no. 2 (1961).

** This question arose in October 1962 during discussion of the work ([6]) at the seminar "Algorithms and Automata" at Novosibirsk State University.

Fix an alphabet M consisting of four letters, and an alphabet M' consisting of 8 letters, so that the alphabet M' is an extension of the alphabet M ($M' \supset M$). We shall call any infinite sequence of letters of the alphabet M a **signifying sequence**. To an arbitrary system of pairs of words S in the alphabet M one can assign a noninitial M'-basis S_A in such a way that the following holds.

Lemma 1. *In the M'-basis S_A one can realize the constant M'-operator $*$, which produces a signifying sequence if and only if the system of pairs of words S is compatible.*

Let A be some finite M'-automaton with input channels $X_1, X_2, \dots, X_p$ and internal states $\{q_1, q_2, \dots, q_k\}$. Associate with it the so-called coded automaton A_{sh} with input channels $X_1, X_2, \dots, X_p, X_{p+1}$ and internal states $\{q_1, q_2, \dots, q_k, q_{k+1}\}$, operating as follows. So long as a sequence which is the beginning of a signifying sequence is supplied to the input X_{p+1} of the automaton A_{sh}, it functions, with respect to the input channels $X_1, X_2, \dots, X_p$, in the same way as the automaton A. Starting from the point at which this condition is first violated, the automaton A_{sh} passes into the state q_{k+1}, in which it outputs a certain letter not belonging to the alphabet M, and from which it can pass only into the same state q_{k+1}.

Lemma 2. *Suppose that in the basis S_A one cannot realize the constant M'-operator that produces a signifying sequence. Then, for any M'-automaton R, in the basis $S_A \cup R_{\text{sh}}$ one also cannot realize such a constant operator.*

The proof of Theorem 1 is easily obtained from Lemmas 1 and 2. The transition from M'-bases to arbitrary bases is achieved by means of a suitable coding of the letters of the alphabet M'.

In conclusion, we note that Theorems 1–3 have been established by us for bases containing elements in which the number of input channels is not greater than three.

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* Recall that an operator is called **constant** if it transforms every input sequence into one and the same output sequence, which, as is known, is necessarily periodic.

Note: Figure translations are in progress. See original paper for figures.