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Abstract

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ON THE BEHAVIOR OF SOLUTIONS OF THE FIRST BOUNDARY-VALUE PROBLEM FOR SOME QUASILINEAR PARABOLIC EQUATIONS

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In the works of G. I. Barenblatt and other authors, properties of solutions of the equation of nonstationary filtration were studied, mainly in connection with problems of mechanics and physics. Degenerate quasilinear parabolic equations were considered in the work ([1,11]) in connection with the investigation of a system of equations in boundary-layer theory. In the review article ([6]), a definition was given of a generalized solution of the first and second boundary-value problems and of the Cauchy problem for a one-dimensional degenerate quasilinear parabolic equation of nonstationary filtration type. Existence and uniqueness theorems for solutions of these problems were formulated and proved, and a qualitative investigation of these solutions was carried out. In subsequent works ([8–10]) and others, solutions of one-dimensional and multidimensional degenerate parabolic equations were considered and their qualitative investigation was carried out. In particular, in ([8]) the asymptotic behavior as $t \to \infty$ of the solution of the first and second boundary-value problems and of the Cauchy problem for the equation of one-dimensional nonstationary filtration was studied.

Self-similar solutions of the first boundary-value problem with zero boundary conditions* for the equation of nonstationary filtration were written down in ([2,3]) and other works, and their asymptotic properties were studied there as well.

In the present work, estimates from above, uniform in ε, are derived for the modulus of the solution $u(x,t;\varepsilon)$ of problem (*) for equation (1) (one-dimensional) and equation (15) (multidimensional). The solution $u(x,t;\varepsilon)$ is assumed a priori to exist. The basic inequalities (6) and (16) are valid for generalized solutions $u(x,t) \equiv u(x,t,0)$ of degenerate quasilinear parabolic equations, which are obtained as the limit of classical solutions of equations (1) and (15) as $\varepsilon \to 0$. In proving inequalities (6) and (16), a technique is used which was applied by E. M. Landis in the qualitative theory of elliptic and parabolic equations of second order (see, for example, ([5,7])).

1. Consider in the domain

$$G_T = \{(x,t) \mid \varphi_1(t) < x < \varphi_2(t);\; 0 < t < T\}$$

the solution $u(x,t;\varepsilon)$ of problem (*) for the equation

$$\mathcal{L}_\varepsilon u(x,t;\varepsilon) - \frac{\partial u(x,t;\varepsilon)}{\partial t} = 0, \tag{1}$$

where

$$\mathcal{L}_\varepsilon u(x,t;\varepsilon) \equiv \frac{\partial}{\partial x}\left\{\left[a\left(x,t,u,\frac{\partial u}{\partial x}\right) + \varepsilon\left(\left(\frac{\partial u}{\partial x}\right)^2 + 1\right)^{(\nu-2)/2}\right]\frac{\partial u}{\partial x}\right\}, \tag{2}$$

$$\frac{\partial a(x,t,u,v)}{\partial v}\, v \geqslant 0, \tag{3}$$

$$a(x,t,u,v) \geqslant a_0(u)|v|^\beta, \tag{4}$$

$$\nu \geqslant 2, \quad \beta \geqslant 0; \qquad a_0(\lambda) > 0,\ \lambda \neq 0; \qquad a_0(0) = 0.$$

Let in G_T

$$u(x,t;\varepsilon) \in C^{(2)} \text{ in } x \text{ and } t, \qquad |u(x,t;\varepsilon)| \leqslant M. \tag{5}$$

Introduce the notation

$$m(\varepsilon) = m(T;\varepsilon) = \max_{(x,T)\in\overline{G}T} |u(x,T;\varepsilon)|, \qquad M = \max_{(x,0)\in\overline{G}T} |u(x,0;\varepsilon)|.$$

Under the assumptions made, the estimate

$$a_0(\xi_0)(m(\varepsilon))^{\beta+1} \leq \frac{(\mu_2 G_T)^{\beta+1}}{T^{\beta+2}}\, CM, \tag{6}$$

is valid, where

$$C = 2^{\beta+1}(\varphi_2(0) - \varphi_1(0)), \qquad m(\varepsilon)/2 < \xi_0 < m(\varepsilon).$$

Obviously, if

$$(\mu_2 G_T)^{\beta+1}/T^{\beta+2} \underset{T}{\rightarrow} 0,$$

then $m(T;\varepsilon) \to 0$ as $T \to \infty$.

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* In what follows, for brevity, instead of "the first boundary-value problem with zero boundary conditions" we shall say "problem (*)" .

Estimate (6) is uniform with respect to ε. It is valid for generalized solutions $u(x,t)$ of the equation $\mathcal{L}_0 u(x,t) - \partial u(x,t)/\partial t = 0$, which are limits of solutions $u(x,t;\varepsilon)$ of equation (1) as $\varepsilon \to 0$.

Let us prove inequality (6). Without loss of generality, we shall assume that $u(x,t;\varepsilon) > 0$ in G_T. Put

$$\Pi_T = \{(x,t) \mid -\infty < x < \infty;\ 0 < t < T\}, \qquad \dot{G}_T = (\overline{G}_T \setminus G_T) \cap \Pi_T,$$

$$l_\xi = \{(x,t) \mid (x,t) \in G_T;\ u(x,t;\varepsilon) = \xi\},$$

$$G_T^\xi = \{(x,t) \mid (x,t) \in G_T;\ u(x,t;\varepsilon) > \xi\},$$

$$N = \{\xi \mid \xi \in [m(\varepsilon)/2, m(\varepsilon)]\ \forall (x,t) \mid (x,t) \in l_\xi\ \nabla u \neq 0\}.$$

Let $\xi \in N$; then all components of l_ξ joining the lower and upper bases of G_T are projected onto the t-axis once. Let

$$\dot{l}_\xi = \{(x,t) \mid (x,t) \in l_\xi,\ \partial u/\partial x \geq 0\},$$

$$\ddot{l}_\xi = \{(x,t) \mid (x,t) \in l_\xi,\ \partial u/\partial x \leq 0\};$$

then, obviously, $l_\xi = \dot{l}_\xi \cup \ddot{l}_\xi$, and the sets $\dot{l}_\xi$ and $\ddot{l}_\xi$ have at least one component each joining the upper and lower bases of the domain G_T.

The inequalities

$$\mu_2 G_T > \int_{G_T^{m(\varepsilon)/2} \setminus G_T^{m(\varepsilon)}} dt\, dx \geq \int_N \int_{l_\xi} \frac{dt}{|\partial u/\partial x|}\, d\xi \geq \frac{m(\varepsilon)}{2} \int_{\dot{l}_{\xi_0}} \frac{dt}{|\partial u/\partial x|} > \frac{m(\varepsilon)}{2} \int_{\dot{l}_{\xi_0}} \frac{dt}{\partial u/\partial x}. \tag{7}$$

are valid.

Using (7) and the Cauchy–Bunyakovsky inequality, we obtain

$$\int_{\dot{l}_{\xi_0}} \frac{\partial u}{\partial x}\, dt > \frac{T^2 m(\varepsilon)}{2\mu_2 G_T}. \tag{8}$$

Consider the integral

$$0=\int_{G_T^{\xi_0}}\left[-\frac{\partial}{\partial x}\left(a\left(x,t,u,\frac{\partial u}{\partial x}\right)+\varepsilon\left(\left(\frac{\partial u}{\partial x}\right)^2+1\right)^{(\nu-2)/2}\frac{\partial u}{\partial x}\right)+\frac{\partial u}{\partial t}\right]dt\,dx=\Phi(\xi_0).$$

Applying Green’ s formula, we obtain

$$\int_{G_T^{\xi_0}}\frac{\partial u}{\partial t}\,dt\,dx=-\int_{\overline{G}_T^{\xi_0}\setminus G_T^{\xi_0}}u\,dx\geq-\int_{\gamma_{\xi_0}}(u-\xi_0)\,dx>-M(\varphi_2(0)-\varphi_1(0))\quad(9)$$

$$(\gamma_{\xi_0}=\overline{G}_T^{\xi_0}\cap\{t=0\});$$

$$-\int_{G_T^{\xi_0}}\frac{\partial}{\partial x}\left(a\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}\right)dt\,dx-\varepsilon\int_{G_T^{\xi_0}}\frac{\partial}{\partial x}\left(\left(\left(\frac{\partial u}{\partial x}\right)^2+1\right)^{(\nu-2)/2}\frac{\partial u}{\partial x}\right)dt\,dx=$$

$$=-\int_{\overline{G}_T^{\xi_0}\setminus G_T^{\xi_0}}a\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}\,dt-\varepsilon\int_{\overline{G}_T^{\xi_0}\setminus G_T^{\xi_0}}\left(\left(\frac{\partial u}{\partial x}\right)^2+1\right)^{(\nu-2)/2}\frac{\partial u}{\partial x}\,dt\geq$$

$$\geq-\int_{l_{\xi_0}}a\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}\,dt=\int_{l_{\xi_0}}a\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}\,dt\geq$$

$$\geq\int_{l_{\xi_0}}a_0(u)\left|\frac{\partial u}{\partial x}\right|^{\beta}\frac{\partial u}{\partial x}\,dt=a_0(\xi_0)\int_{l_{\xi_0}}\left(\frac{\partial u}{\partial x}\right)^{\beta+1}dt.\qquad(10)$$

By Hölder’ s inequality,

$$\int_{l_{\xi_0}}\frac{\partial u}{\partial x}\,dt\leq\left(\int_{l_{\xi_0}}\left(\frac{\partial u}{\partial x}\right)^{\beta+1}dt\right)^{1/(\beta+1)}\left(\int_{l_{\xi_0}}dt\right)^{\beta/(\beta+1)}=\left(\int_{l_{\xi_0}}\left(\frac{\partial u}{\partial x}\right)^{\beta+1}dt\right)^{1/(\beta+1)}T^{\beta/(\beta+1)},$$

whence

$$\int_{l_{\xi_0}}\left(\frac{\partial u}{\partial x}\right)^{\beta+1}dt\geq\frac{1}{T^{\beta}}\left(\int_{l_{\xi_0}}\frac{\partial u}{\partial x}\,dt\right)^{\beta+1}.\qquad(11)$$

Combining (10), (11), (9), (8), we obtain

$$0 = \Phi(\xi_0) \geq a_0(\xi_0)\frac{T^{\beta+2}(m(\varepsilon))^{\beta+1}}{2^{\beta+1}(\mu_2 G_T)^{\beta+1}} - M(\varphi_2(0) - \varphi_1(0)),$$

whence inequality (6) follows immediately.

2. Let us consider some particular cases. Suppose in (2) $v = 2$, $a(x, t, u, \partial u/\partial x) = u^\alpha$, $\alpha > 0$, and $\varphi_i = (-1)^i k(t+h)^\omega$ $(i = 1, 2)$, where k, h, ω are some positive constants; then $\mu_2 G_T = \dfrac{2k}{\omega+1}((T+h)^{\omega+1} - h^{\omega+1})$, $\mathcal{L}_0 u(x,t) = \dfrac{\partial}{\partial x}\left(u^\alpha \dfrac{\partial u}{\partial x}\right)$, and (1) for $\varepsilon = 0$ becomes the equation of nonstationary filtration. For a generalized solution $u(x,t)$ of problem (*) for this equation, estimate (6) has the form

$$m^{\alpha+1} \leq C_1 M / T^{1-\omega}, \qquad C_1 = C(\alpha, k, h). \tag{12}$$

Put $\omega = 1/(\alpha+2)$; then from (12) we obtain

$$m < C_2 M^{1/(\alpha+1)} / T^{1/(\alpha+2)}. \tag{13}$$

In [3] a self-similar solution $u_0(x,t)$ of the equation of nonstationary filtration is written down, for which

$$\max_{(x,T)\in\overline{G}_T} u_0(x,T) = \frac{C^*}{(T+\theta)^{1/(\alpha+2)}} \qquad (C^*, \theta \text{ are positive constants}).$$

For equation (1) in

$$G_T^* = \{(x,t) \mid -k(t+h)^\omega < x < k(t+h)^\omega,\ 0 < t < T\}$$

estimate (6) has the form

$$a_0(\xi_0) m^{\beta+1} < C_3 M / T^{1-\omega(\beta+1)}. \tag{14}$$

If $\omega < 1/(1+\beta)$, then $1 - \omega(1+\beta) > 0$, and hence $m(T; \varepsilon) \to 0$ as $T \to \infty$.

3. Consider in the domain

$$G^T = \{(x,t) \mid 0 \leq r < t^{\omega/2},\ 1 < t < T\}, \qquad r = |x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2},$$

the solution $u(x, t; \varepsilon)$ of problem (*) for the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x,t,u) \frac{\partial u}{\partial x_j} \right) + \varepsilon \Delta u - \frac{\partial u}{\partial t} = 0. \tag{15}$$

Suppose that in G^T

$$\sum_{i,j=1}^{n} a_{ij}(x,t,u)\eta_i\eta_j \geq a_0(u) \sum_{i=1}^{n} \eta_i^2; \qquad a_0(\lambda) > 0, \quad \lambda \neq 0,$$

$$a_0(0) = 0.$$

Put

$$\max_{(x,1)} |u(x,1;\varepsilon)| = M, \qquad \max_{(x,T)} |u(x,T;\varepsilon)| = m(T;\varepsilon) = m(\varepsilon).$$

Under the assumptions made, the estimate is valid

$$a_0(\xi_0) m(\varepsilon) < C_4 M / T^{(1-\omega)/2}, \qquad C_4 = C(n). \tag{16}$$

If $\omega < 1$, then $1 - \omega = \sigma > 0$, and hence $m(T;\varepsilon) \to 0$ as $T \to \infty$. Estimate (16) is uniform with respect to ε. It is valid for generalized solutions.

$u(x,t)$ of the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x,t,u) \frac{\partial u}{\partial x_j} \right) - \frac{\partial u}{\partial t} = 0,$$

which are limits of solutions $u(x,t;\varepsilon)$ of equation (15) as $\varepsilon \to 0$. The proof of estimate (16) is, in idea, no different from the proof of estimate (6), but is longer. It is close to the proof of Lemma 1 in paper [5].

4. Remark 1. Estimate (6) can be obtained for the case when

$$\mathcal{L}_0 u(x,t) \equiv \frac{\partial}{\partial x} \left(a \left(t, u, \frac{\partial u}{\partial x} \right) \right),$$

where

$$a(t,u,v) \geq |v|^{1+\beta} a_0(u) \quad (\beta \geq 0), \qquad \frac{da(t,u,v)}{dv} \geq 0.$$

Remark 2. For the solution $u(x,t,\varepsilon)$ of problem $(*)$ for the equation

$$\mathcal{L}_\varepsilon u(x,t;\varepsilon) - \partial(b(u)+\varepsilon u)/\partial t = 0;$$

$$b(\lambda) > 0, \quad b'(\lambda) > 0, \quad \lambda \neq 0, \qquad b(0) = b'(0) = 0,$$

one can obtain estimate (6′), in which $b(M)$ will appear instead of M.

Remark 3. In paper [10] it is shown that the generalized solution $u(x,t)$ ($0 \leq u(x,t) \leq M$) of problem (∗) for the equation

$$\partial^2 u/\partial x^2 - b(u)\,\partial u/\partial t = 0, \qquad b(\lambda) > 0, \quad b'(\lambda) > 0, \quad \lambda > 0, \quad b(0) = 0,$$

in $G = \{(x,t) \mid 0 < x < l,\ t > 0\}$ is finite with respect to t if and only if

$$\int_0^\infty \frac{b(u)}{u}\,du < \infty.$$

If

$$\int_0^\infty \frac{b(u)}{u}\,du = \infty,$$

then the solution $u(x,t)$ is not finite with respect to t, but then, as is easy to show, it must decrease no more slowly than $2M\exp(-Ht)$, where $H > 0$ is a constant depending on l and $b(M)$. This is also true for the solution $u(x,t)$ of problem (∗) for the equation $\mathcal{L}u - b(u)\,\partial u/\partial t = 0$, where $\mathcal{L}u$ is an elliptic operator (with a nonzero ellipticity constant) for which the maximum principle holds.

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Note: Figure translations are in progress. See original paper for figures.