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N. G. ASKEROV, S. G. KREIN, G. I. LAPTEV

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Abstract

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N. G. ASKEROV, S. G. KREIN, G. I. LAPTEV

ON A CLASS OF NON-SELF-ADJOINT BOUNDARY-VALUE PROBLEMS

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A number of problems of mathematical physics lead to homogeneous boundary-value problems containing one and the same parameter λ in the differential equations and in the boundary conditions. Despite the fact that, for each fixed λ, the differential operator and the boundary conditions are self-adjoint, the problem as a whole is often non-self-adjoint; the spectrum may be non-real. In the present article a general consideration of one class of such problems is carried out.

1°. Let, in a separable Hilbert space H with scalar product $(\, , \,)$, a linear operator A with everywhere dense domain of definition $D(A)$ be given. Suppose, moreover, that on $D(A)$ two linear operators T and Γ are defined, mapping $D(A)$ into some other separable Hilbert space H_1 with scalar product $(\, , \,)_1$. The operators A, T, Γ have the following properties:

1) The set of elements of $D(A)$ satisfying the conditions $\Gamma v = 0$ and $Tv = 0$ is dense in H.
2) The restriction A_0 of the operator A to the set of all elements of $D(A)$ for which $Tu = 0$ is a self-adjoint, positive-definite operator having a completely continuous inverse.

We introduce into consideration the operator $A_0^{1/2}$. Its domain of definition $D(A_0^{1/2})$ will be regarded as the Hilbert space $H_{1/2}$ with scalar product

$$[uv] = (A_0^{1/2}u, A_0^{1/2}v) \quad (u, v \in D(A_0^{1/2})).$$

As is known, the space $H_{1/2}$ can be obtained from $D(A_0)$ by completing it with respect to the norm

$$\sqrt{[u,u]} = \sqrt{(A_0u,u)}.$$

3) The operator Γ maps $D(A_0)$ onto a set dense in H_1, and at the same time is completely continuous as an operator from the space $H_{1/2}$ into the space H_1. We shall regard the operator Γ as extended by continuity to the whole space $H_{1/2}$; moreover

$$\|\Gamma u\|_1 \le C\|A_0^{1/2}u\| \quad (u \in D(A_0^{1/2}) = H_{1/2}).$$

4) The "Green formula" holds:

$$(Au, v) = A(u, v) - (Tu, \Gamma v)_1, \tag{1}$$

where $A(u, v)$ is a bilinear functional such that $A(u, u) \geq 0$. It is not difficult to see that the functional $A(u, v)$ can be extended by continuity to all elements $u, v \in D(A_0^{1/2})$, and moreover

$$A(u, v) = (A_0^{1/2} u, A_0^{1/2} v).$$

Denote by N the set of all elements w in $D(A_0^{1/2})$ for which the inequality

$$|(A_0^{1/2} w, A_0^{1/2} z)| \leq C_w \|\Gamma z\|_1$$

is valid for any $z \in D(A_0^{1/2})$.

Lemma 1. For each $\varphi \in H_1$ there exists a unique element $w \in N$ satisfying the identity

$$(A_0^{1/2} w, A_0^{1/2} z) = (\varphi, \Gamma z)_1 \tag{2}$$

for any $z \in D(A_0^{1/2})$. Conversely, to each $w \in N$ there corresponds a unique element $\varphi \in H_1$ satisfying identity (2).

The lemma is proved by ordinary arguments with the aid of Riesz' s theorem on the general form of a linear functional in Hilbert space.

If $w \in N \cap D(A)$, then the conditions $Aw = 0$ and $Tw = \varphi$ are satisfied. Consequently, the elements $w \in N$ may be regarded as generalized solutions of these equations. In what follows, for any $w \in N$ we shall denote $\varphi = Tw$ and $w = T^{-1}\varphi$.

Lemma 2. *The operator* $R = A_0^{1/2} T^{-1} \Gamma A_0^{-1/2}$ *is a self-adjoint nonnegative completely continuous operator in* H.

2°. **Statement of the problem:** it is required to find a solution of the equation

$$Ay = \lambda y, \tag{3}$$

satisfying the condition

$$\lambda T y = \sigma \Gamma y, \tag{4}$$

where the given $\sigma > 0$.

By a **generalized solution** of problem (3)–(4) we shall mean an element $y \in D(A_0^{1/2})$, representable in the form $y = u + w$, where $u \in D(A_0)$, $w \in N$ satisfy the equations $A_0 u = \lambda y$ and $\lambda T w = \sigma \Gamma y$.

Theorem 1. *The problem of finding generalized solutions of (3)–(4) is equivalent to the problem of solving, in the space H, the equation*

$$x = \lambda A_0^{-1} x + \frac{\sigma}{\lambda} R x. \tag{5}$$

If x is a solution of (5), then $y = A_0^{1/2} x$ is a generalized solution of (3)–(4), and conversely.

3°. Let us investigate in a general form, in the Hilbert space H, the equation

$$f = \lambda P f + \frac{1}{\lambda} Q f, \tag{6}$$

where P is a positive and Q a nonnegative completely continuous operator in H.

From the results of I. Ts. Gokhberg ([1]) (see also ([2])) it follows that the numbers λ corresponding to nonzero solutions of equation (6) (eigenvalues) can have only two points of accumulation: $\lambda = 0$ and $\lambda = \infty$. In this connection it is natural to try to transform equation (6) to such a form that the eigenvalues of the new equation have only one point of accumulation, $\lambda = 0$.

It is verified directly that equation (6) is equivalent to the system of equations

$$P^{1/2} B P^{1/2} g + P^{1/2} B Q^{1/2} h = \frac{1}{1+\lambda} g,$$

$$-Q^{1/2} B P^{1/2} g + (I - Q^{1/2} B Q^{1/2}) h = \frac{1}{1+\lambda} g, \tag{7}$$

where $g = P^{1/2} f$, $h = \dfrac{1}{\lambda} Q^{1/2} f$, and $B = (I + P + Q)^{-1}$.

System (7) may be regarded as the equation

$$\mathfrak{A} X = \mu X \qquad \left(X = \begin{pmatrix} g \\ h \end{pmatrix}, \quad \mu = \frac{1}{1+\lambda} \right)$$

in the space $H \times H$ with the operator

$$\mathfrak{A} = \begin{pmatrix} P^{1/2} B P^{1/2} & P^{1/2} B Q^{1/2} \\ -Q^{1/2} B P^{1/2} & I - Q^{1/2} B Q^{1/2} \end{pmatrix}.$$

If $\lambda = 0$ and $\lambda = \infty$ were points of accumulation of the eigenvalues of equation (6), then the points of accumulation of the spectrum of the operator $\mathfrak{A}$ will be

$\mu = 1$ and $\mu = 0$. Construct the operator $\mathfrak{B} = \mathfrak{A}(I - \mathfrak{A})$. By direct calculation we obtain

$$\mathfrak{B} = \begin{pmatrix} P^{1/2}B(I+2Q)BP^{1/2} & -P^{1/2}B(P-Q)BQ^{1/2} \\ Q^{1/2}B(P-Q)BP^{1/2} & Q^{1/2}B(I+2P)BQ^{1/2} \end{pmatrix}. \tag{8}$$

The operator $\mathfrak{B}$ turns out to be completely continuous. Between the eigenvalues λ of equation (6) and the eigenvalues ν of the operator $\mathfrak{B}$ there is the relation $\lambda/(1+\lambda)^2 = \nu$.

Theorem 2. *All eigenvalues of equation (6) have nonnegative real part. If the condition*

$$4\|P\|\,\|Q\| \le 1, \tag{9}$$

is satisfied, then all eigenvalues are real. If condition (9) is not satisfied, then the nonreal eigenvalues of equation (6) can be located only in the annulus $1/2\|P\| \le |\lambda| \le 2\|Q\|$ *and, consequently, they form a finite set.*

If we consider the sum of the root subspaces of the operator $\mathfrak{B}$ corresponding to a given eigenvalue ν, then we obtain an invariant finite-dimensional subspace of the operator $\mathfrak{A}$. In it one can choose a basis in which the operator $\mathfrak{A}$ has normal Jordan form. In the subspace corresponding to one Jordan cell, it will be convenient for us to choose a basis consisting of elements $X_0, X_1, \dots, X_m$ such that

$$\mathfrak{A}X_k = \sum_{l=0}^{k} \frac{(-1)^{k-l}}{(1+\lambda)^{k+1-l}} X_l \quad (k = 0, 1, \dots, m).$$

If $X_k = \binom{g_k}{h_k}$, then the elements $f_k = P^{-1/2} g_k$ satisfy the equations

$$f_k = \lambda P f_k + \frac{1}{\lambda} Q f_k + P f_{k-1} + \sum_{l=0}^{k-1} \frac{(-1)^{k-l}}{\lambda^{k+1-l}} Q f_l \quad (k = 0, 1, \dots, m).$$

These equations can be obtained formally by differentiating equation (6) with respect to λ, if one sets $f_k = \frac{1}{k!}\frac{\partial^k f_0}{\partial \lambda^k}$. The subspace spanned by each chain of elements $\{f_0, f_1, \dots, f_m\}$ of maximal length is called the **root subspace** corresponding to the eigenvalue λ.

The real part of the operator $\mathfrak{B}$ is determined by the diagonal entries of the matrix (8) and is a nonnegative operator; therefore, for the study of the question of completeness of the system of root subspaces of equation (6), one can apply the theory of dissipative operators. If the operators P and Q have finite trace, then all operators of the matrix (8) also have finite trace. With the aid of results of V. B. Lidskii ([3]) and M. G. Krein ([4]) one can obtain:

Theorem 3. *If the operators P and Q have finite trace, then the system of root subspaces of equation (6) is doubly complete in the space H in the following sense (see ([5])): for every pair of elements $x, y \in H$ there exists a sequence of linear combinations $\sum c_{sk}^{(N)} f_k^{(s)}$ $(N = 1, 2, \dots)$ of eigen and associated elements of equation (6) (s is the number of the root subspace, k is the number of the associated element) such that, as $N \to \infty$,*

$$\sum_k c_{sk}^{(N)} f_k^{(s)} \to x \quad \text{in the norm } (Px, x),$$

$$\sum c_{sk}^{(N)} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{\lambda^{k+1-j}} f_j^{(s)} \to y \quad \text{in the seminorm } (Qx, x).$$

* Condition (9) can be replaced by the less restrictive condition $4(Pf, f)(Qf, f) \leq (f, f)^2$ for all $f \in H$.

** As M. G. Krein informed us, results obtained by him jointly with G. Langer make it possible in some cases to extend Theorem 3.

4°. The elements $y_1, y_2, \dots, y_m$ will be called **associated** with the generalized solution y_0 of problem (3)–(4) if $y_k = u_k + w_k$, where $u_k \in D(A_0)$, $w_k \in N$, and they satisfy the equations

$$A_0 u_k = \lambda y_k + y_{k-1}, \qquad \lambda T w_k + T w_{k-1} = \sigma \Gamma y_k \quad (k = 1, 2, \dots, m).$$

The subspace spanned by $y_0, y_1, \dots, y_m$ is called the **root subspace** of problem (3)–(4).

Denote by $\{e_k\}$ the complete orthonormal system of eigenvectors of the operator A_0, and by $\{\mu_k\}$ the system of corresponding eigenvalues. Then, in the basis $\{e_k\}$, the operator R is given by the matrix

$$R = \left(\frac{(\Gamma e_i, \Gamma e_j)}{\sqrt{\mu_i}\sqrt{\mu_j}} \right).$$

Its trace is equal to $\sum \frac{\|\Gamma e_n\|_1^2}{\mu_n}$. From Theorems 1–3 there follows

Theorem 4. *Starting from some index, all eigenvalues of problem (3)–(4) are real. If the conditions*

$$\sum \frac{1}{\mu_n} < \infty, \qquad \sum \frac{\|\Gamma e_n\|_1^2}{\mu_n} < \infty, \tag{10}$$

are satisfied, then the system $\{y_k^{(s)}\}$ of generalized and associated solutions of problem (3)–(4) is doubly complete in the following sense: for each pair of elements $x \in H$ and $\varphi \in H_1$ there are linear combinations of solutions $y_k^{(s)}$ such that, as $N \to \infty$,

$$\sum c_{sk}^{(N)} y_k^{(s)} \longrightarrow x \quad \text{in } H; \qquad \sum c_{sk}^{(N)} \sum_{j=0}^{k} \frac{(-1)^{k-j}\Gamma y_j^{(s)}}{\lambda^{k+1-j}} \longrightarrow \varphi \quad \text{in } H_1. \tag{11}$$

Remark 1. Theorem 4 remains valid if the multiplier σ is replaced by a bounded nonnegative operator in H_1. In the last part of the theorem one can then assert that the linear combinations (11) converge in the seminorm $(\sigma\varphi, \varphi)_1$.

Remark 2. If σ is negative, then problem (3)–(4) is also reduced to the investigation of equation (5), or to the equation

$$f = \lambda P f - \frac{1}{\lambda} Q f,$$

where P and Q are the same as in (6). The last equation, by the substitution $g = P^{1/2} f$ and $h = \frac{1}{\lambda} Q^{1/2} f$, is reduced to the equation $\mathfrak{A}X = \lambda X$, where $\mathfrak{A}$ is a self-adjoint operator in the space $H \times H$. Thus, for $\sigma < 0$ the problem is sharply simplified. For any pair of elements $x \in H$ and $\varphi \in H_1$, one can write expansions in series in the generalized solutions of problem (3)–(4).

Remark 3. The conditions under which the first part of Theorem 4 has been proved are verified in various problems for partial differential equations with the help of embedding theorems and energy inequalities. In this case the operator A is defined by a differential expression, while the operators T and Γ are boundary operators. Conditions (10) are considerably more restrictive. To broaden the range of possible applications, the conditions on the operators P and Q in Theorem 3 should be weakened.

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Note: Figure translations are in progress. See original paper for figures.