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Abstract

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ANALYTIC MAPPINGS AND THE BERGMAN FUNCTION

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Let G be an arbitrary bounded domain in the space C^n of complex variables $z_1, \dots, z_n$. By $L(G, z^0)$ $(z^0 \in G)$ denote the set of all analytic functions $f(z) = f(z_1, \dots, z_n)$, regular and single-valued in G and, moreover, satisfying the condition $|f(z^0)| = 1$. The Bergman function $K_G(z^0; \bar{z}^0)$ of the domain G at the point z^0, as is known, is defined by the equality

$$\{K_G(z^0; \bar{z}^0)\}^{-1} = \inf_{f \in L(G, z^0)} A_G[f(z)],$$

where

$$A_G[f(z)] = \int_G |f(z)|^2 \, d\omega.$$

Every collection $(\varphi_1, \dots, \varphi_n)$ of n analytic functions $\varphi_k(z)$ $(k = 1, 2, \dots, n)$, regular in the domain G (possibly multiple-valued), such that the function

$$J(z, \varphi_1, \dots, \varphi_n) = \frac{\partial(\varphi_1, \dots, \varphi_n)}{\partial(z_1, \dots, z_n)}$$

is regular and single-valued in G, will be called an **analytic mapping** of the domain G. The analytic mapping $(\varphi_1, \dots, \varphi_n)$ is called **normalized** at the point $z^0 \in G$ if $|J(z^0, \varphi_1, \dots, \varphi_n)| = 1$. By $M(G, z^0)$ denote the set of all analytic mappings of the domain G normalized at the point $z^0 \in G$. Put

$$\{P_G(z^0)\}^{-1} = \inf_{(\varphi_1, \dots, \varphi_n) \in M(G, z^0)} A_G[J(z, \varphi_1, \dots, \varphi_n)].$$

Theorem 1. For every point $z^0 \in G$ the equality

$$K_G(z^0; \bar{z}^0) = P_G(z^0)$$

holds.

Denote by $E(G)$ the set of those points $z^0 \in G$ for which the following is fulfilled: whatever analytic mapping $(\varphi_1, \dots, \varphi_n) \in M(G, z^0)$ may be, one always has $A_G[J(z, \varphi_1, \dots, \varphi_n)] \geq V(G)$, where $V(G) = A_G$ is the volume of the domain G (1).

Theorem 2. In order that a point $z^0 \in E(G)$, it is necessary and sufficient that the equality

$$K_G(z^0; \bar{z}^0) = \{V(G)\}^{-1}$$

hold.

Proof. Suppose that $K_G(z^0; \bar{z}^0) = \{V(G)\}^{-1}$. If an analytic mapping $(\varphi_1, \dots, \varphi_n) \in M(G, z^0)$, then

$$A_G[J(z, \varphi_1, \dots, \varphi_n)] \geq \{P_G(z^0)\}^{-1} = \{K_G(z^0; \bar{z}^0)\}^{-1} = V(G).$$

Let $z_0 \in E(G)$. Suppose that $K_G(z^0; \bar{z}^0) > \{V(G)\}^{-1*}$. Then for every analytic mapping $(\varphi_1, \dots, \varphi_n) \in M(G, z^0)$

$$A_G[J(z, \varphi_1, \dots, \varphi_n)] \geq V(G),$$

and, consequently,

$$\{P_G(z^0)\}^{-1} \geq V(G) > \{K_G(z^0; \bar{z}^0)\}^{-1}.$$

The last inequality contradicts Theorem 1.

Theorem 3. Let $G = G_1 \times \dots \times G_n$; G_m $(m = 1, 2, 3, \dots, n)$ are bounded simply connected domains of the space C^1. If the set $E(G)$ contains at least one point $z^0 = (z_1^0, \dots, z_n^0)$, then G is a polycylinder with center at the point z^0, and the set $E(G)$ consists only of the point z^0.

Proof. If one takes into account the equalities

$$V(G) = \prod_{m=1}^{n} V(G_m)$$

and

$$K_G(z^0; \bar{z}^0) = \prod_{m=1}^{n} K_{G_m}(z_m^0; \bar{z}_m^0),$$

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* Note that always $K_G(z^0; \bar{z}^0) \geq \{V(G)\}^{-1}$.

then it is necessary to prove Theorem 3 for $n = 1$. Let the point $z_1^0 \in E(G_1)$, and let $\zeta = \varphi(z)$ be a function satisfying the conditions $\varphi(z_1^0) = 0$, $\varphi'(z_1^0) = 1$, and mapping G_1 one-to-one onto the disk $\Delta_{z_1^0} : |\zeta| < R_{z_1^0}$, where $R_{z_1^0}$ is the conformal radius of the domain G_1 at the point z_1^0. As is known,

$$K_{G_1}(z_1^0; \bar{z}_1^0) = \{\pi R_{z_1^0}^2\}^{-1}$$

([1], p. 90), and consequently

$$V(G_1) = \pi R_{z_1^0}^2.$$

It follows that $\varphi^{-1}(\zeta) = z_1^0 + \zeta$, i.e., the domain G is a disk with center at the point z_1^0 and radius $R_{z_1^0}$. If one takes into account that for the disk $\Delta_1 : |z_1| < R$ the Bergman function is

$$K_{\Delta_1}(z_1^0; \bar{z}_1^0) = \pi^{-1} R^2 (R^2 - |z_1^0|^2)^{-2},$$

one may conclude that the set $E(\Delta_1)$ consists only of the point $z_1^0 = 0$.

Remark. It follows from Theorem 3 that if the domain $G \subset C^n$ is a polycylinder with center at the point z^0, then for every analytic mapping $(\varphi_1, \dots, \varphi_n) \in M(G, z^0)$ the inequality

$$A_G[J(z, \varphi_1, \dots, \varphi_n)] \geq V(G)$$

holds.* Moreover, as is easy to show, the equality sign is possible if and only if

$$|J(z, \varphi_1, \dots, \varphi_n)| \equiv 1.$$

We note that if $n > 1$, it does not follow from the last equality that the analytic mapping $(\varphi_1, \dots, \varphi_n)$ is "linear."

Theorem 4. *Let the functions $\varphi_1, \dots, \varphi_n$ be regular and single-valued in the domain $G \subset C^n$, and let $T = (\varphi_1, \dots, \varphi_n)$ be a one-to-one analytic mapping belonging to the set $M(G, z^0)$. Put $H = T(G)$, and let the point $z^0 \in E(G)$. Then, if $V(H) = V(G)$, the point $Tz^0 = w^0 \in E(H)$.*

Let $T = (\varphi_1, \dots, \varphi_n)$ be a one-to-one analytic mapping of the polycylinder $G \subset C^n$, belonging to the set $M(G, z^0)$, where z^0 is the center of G. Put $H = T(G)$, and suppose that $V(H) = V(G)$. By Theorem 4, the point $w^0 = Tz^0 \in E(H)$. Moreover, if H is a polycylindrical domain, then it will necessarily be a polycylinder (Theorem 3), and then, according to a theorem of H. Cartan ([1], p. 30), the mapping T is linear.

Thus, we have established the following theorem:

Theorem 5. *Let $G \subset C^n$ be a polycylinder with center at the point z^0, and let $T = (\varphi_1, \dots, \varphi_n) \in M(G, z^0)$ be a one-to-one analytic mapping of the domain*

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[1] H. Cartan, *J. Math. pures et appl.*, **10** (1931).

G onto a polycylindrical domain H. If $V(H) = V(G)$, then H is a polycylinder with center at the point $w^0 = Tz^0$, and the mapping T is linear.

Remark. The last assertion, together with the remark to Theorem 3, is in a certain sense an analogue of the so-called inner theorem of areas from the theory of functions of one complex variable. We note that Theorem 5 can be proved by relying only on the above-mentioned theorem of H. Cartan and on the remark to Theorem 3.

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REFERENCES

* This assertion can also be proved directly.

Note: Figure translations are in progress. See original paper for figures.