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# MATHEMATICS

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**Abstract**

**Full Text**

## MATHEMATICS

**R. D. Bantsuri, G. A. Dzhanashia**

### ON CONVOLUTION-TYPE EQUATIONS FOR THE HALF-LINE

*(Presented by Academician N. I. Muskhelishvili, 18 XI 1963)*

1. The homogeneous convolution-type equation for the half-line

$$\varphi(x) - \int_0^{\infty} k(x-y)\varphi(y) dy = 0, \quad 0 \leq x < \infty, \quad (A_0)$$

in the case when the kernel  $k(x)$  decreases exponentially at infinity, and the solution is sought in a class of functions having the corresponding growth at infinity, was first considered in the well-known work of Wiener and Hopf <sup>(1)</sup>. The inhomogeneous equation

$$\varphi(x) - \int_0^{\infty} k(x-y)\varphi(y) dy = f(x), \quad 0 \leq x < \infty, \quad (A)$$

under the conditions of Wiener and Hopf was studied in <sup>(2,3)</sup>.

I. M. Rapoport <sup>(4)</sup> was the first to draw attention to the connection of equation (A) with the Hilbert boundary-value problem <sup>(5)</sup>, p. 146, (adjoint problem), proved Noether-type theorems, and gave a solution  $\varphi(x) \in L_2(0, \infty)$ , in quadratures under the assumption that  $k(x) \in L_{1,2}(-\infty, \infty)$ ;  $f(x) \in L_2(0, \infty)$ ;  $K(t) = \int_{-\infty}^{\infty} k(x)e^{ixt} dx$  (below, capital letters will denote the Fourier transforms of the functions denoted by the corresponding lowercase letters) belongs to the class  $\text{Lip } \alpha$ ,  $\alpha > 0$ , for  $-\infty < t < \infty$ ;  $K(t) = O(|t|^{-\beta})$ ,  $\beta > 0$ , as  $t \rightarrow \infty$ , and  $1 - K(t) \neq 0$ . M. G. Krein <sup>(6)</sup>, developing the Wiener-Hopf method, considered the case when  $k(x) \in L(-\infty, \infty)$ ,  $f(x) \in L_p(0, \infty)$ ,  $1 \leq p < \infty$ ,  $f(x) \in M(0, \infty)$  ( $M(0, \infty)$  is the space of bounded measurable functions), and proved that the condition  $1 - K(t) \neq 0$  is necessary and sufficient for Noether-type theorems to hold.

2. In the present note a solution in quadratures is given for  $\varphi(x) \in L(0, \infty)$  of equation (A), when  $k(x) \in L(-\infty, \infty)$ ,  $1 - K(t) \neq 0$ , and  $f(x) \in L(0, \infty)$ . Following Wiener and Hopf <sup>(1)</sup>, we consider the equation

$$\varphi(x) - \int_{-\infty}^{\infty} k(x-y)\varphi(y) dy = f(x) + b(x), \quad -\infty < x < \infty, \quad (A_b)$$

where it is assumed that

$$b(x) = f(-x) = 0 \quad \text{for } 0 < x < \infty,$$

$$b(x) = - \int_{-\infty}^{\infty} k(x-y)\varphi(y) dy \quad \text{for } -\infty < x < 0.$$

It is easy to see that a solution of equation (A), extended by zero for negative  $x$ , is a solution of equation  $(A_b)$ , and that all solutions of equation  $(A_b)$  are obtained in this way. Denote by  $R_0$  the set of

of all functions

$$\Omega(t) = \int_{-\infty}^{\infty} \omega(x)e^{ixt} dx,$$

where  $\omega(x) \in L(-\infty, \infty)$ .  $R_0$  is a certain ring of continuous functions on the closed line ((7), p. 80). Next we denote by  $R_0^+$  ( $R_0^-$ ) the subring of  $R_0$  consisting of functions  $\Omega_1(t)$  ( $\Omega_2(t)$ ) such that  $\omega_1(x) = 0$  for  $x < 0$  ( $\omega_2(x) = 0$  for  $x > 0$ ). The ring obtained by extending  $R_0$  ( $R_0^+$ ,  $R_0^-$ ) by adjoining constants to it will be denoted by  $R$  ( $R^+$ ,  $R^-$ ).

**Remark 1.** It is easy to see that if  $\Omega_1(t) \in R_0^+$  ( $\Omega_2(t) \in R_0^-$ ), then it is the continuous boundary value of a function analytic in the upper half-plane  $\Pi_+$  (in the lower half-plane  $\Pi_-$ ) and vanishing at infinity in the closed half-plane  $\bar{\Pi}_+$  ( $\bar{\Pi}_-$ ).

Here we formulate the Wiener–Lévy theorem ((8), p. 247) and the Wiener theorem ((9), Ch. IV), which are essential for our purposes.

**Wiener–Lévy theorem.** *Let  $G(z)$  be an analytic function in a domain containing the curve  $\gamma = \Omega(t)$ ,  $\Omega(t) \in R$ ; then  $G(\Omega(t)) \in R$ .*

**Wiener theorem.** *If  $\Omega_1(t) \in R^+$  ( $\Omega_2(t) \in R^-$ ), and  $G_1(z)$  ( $G_2(z)$ ) is an analytic function in a domain containing all values of the function  $\Omega_1(s)$ ,  $s \in \bar{\Pi}_+$  ( $\Omega_2(s)$ ,  $s \in \bar{\Pi}_-$ ), then  $G_1(\Omega_1(t)) \in R^+$  ( $G_2(\Omega_2(t)) \in R^-$ ).*

Applying the Fourier transform to equation  $(A_b)$ , we obtain the following Wiener–Hopf problem for the axis:

$$(1 - K(t))\Phi(t) = F(t) + B(t), \quad -\infty < t < \infty, \quad (\tilde{A}_b)$$

where  $K(t), F(t) \in R_0$ , while  $\Phi(t)$  and  $B(t)$  are the sought functions from  $R_0^+$  and  $R_0^-$ , respectively.

3. Consider the following Hilbert boundary-value problem: find a function  $\Psi^+(z)$ , analytic in  $\Pi_+$ , continuous on  $\bar{\Pi}_+$ , and a function  $\Psi^-(z)$ , analytic in  $\Pi_-$ , continuous on  $\bar{\Pi}_-$ , which vanish at infinity and satisfy on the axis the relation

$$(1 - K(t))\Psi^+(t) = \Psi^-(t) + F(t), \quad -\infty < t < \infty, \quad (H)$$

where the functions  $K(t)$  and  $F(t)$  are the same as in problem  $(\tilde{A}_b)$ .

It follows from Remark 1 that the solution of problem  $(\tilde{A}_b)$  is the boundary value of the solution of problem  $(H)$ . We shall solve problem  $(H)$  and show the converse. Using the known properties of the Cauchy-type integral ((<sup>10</sup>), Ch. III), it is easy to prove the following lemma.

**Lemma.** For any  $\Omega(t) \in R_0$  and for any  $t_0 \in (-\infty, \infty)$ , the equality

$$\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\Omega(t)}{t - t_0} dt = \int_{-\infty}^{\infty} \omega(x) \operatorname{sign} x \cdot e^{ixt_0} dx,$$

holds, where the integral on the left is understood in the sense of the principal value at the point  $t_0$  and at infinity (in general, the Cauchy-type integrals occurring below are understood in the sense of the principal value at infinity).

The Wiener–Lévy theorem, the Wiener theorem, and the lemma stated above allow one to solve the Hilbert problem  $(H)$  in the usual way.

4. If the index of the Hilbert problem  $(H)$ , which is naturally called the index of equation  $(A)$ ,

$$\varkappa = -\operatorname{ind}(1 - K(t)) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d \arg(1 - K(t)) \geq 0,$$

then the boundary values of the general solution of problem  $(H)$  have the form

$$\Psi^+(t) = \frac{1}{2} X^+(t) \left( \frac{F(t)}{(1 - K(t))X^+(t)} + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{F(t_0) dt_0}{X^+(t_0)(1 - K(t_0))(t_0 - t)} + \frac{X^+(t)P_{\varkappa-1}(t)}{(t+i)^\varkappa} \right), \quad (1)$$

$$\Psi^-(t) = \frac{1}{2} X^+(t) \left( \frac{t+i}{t-i} \right)^\varkappa \left( -\frac{F(t)}{(1 - K(t))X^+(t)} + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{F(t_0) dt_0}{X^+(t_0)(1 - K(t_0))(t_0 - t)} + \frac{X^-(t_0)P_{\varkappa-1}(t)}{(t-i)^\varkappa} \right),$$

where

$$X^\pm(t) = \exp \left( \mp \frac{1}{2} \ln \left( \frac{t+i}{t-i} \right)^\varkappa (1 - K(t)) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln \left( \frac{t_0+i}{t_0-i} \right) (1 - K(t_0))}{t_0 - t} dt_0 \right);$$

$P_{\varkappa-1}(t)$  is an arbitrary polynomial of degree  $\leq \varkappa - 1$ , if  $\varkappa > 0$ ;  $P_{\varkappa-1}(t) \equiv 0$ , if  $\varkappa = 0$ .

On the basis of Wiener's theorem and of the fact that  $\frac{1}{t \pm i} \in R_0^\pm$ , it is easy to see that  $\Psi^\pm(t) \in R_0^\pm$ ; consequently, (1) is a solution of problem  $(\tilde{A}_b)$ .

Let the index  $\varkappa < 0$ ; the condition

$$\int_{-\infty}^{\infty} \frac{F(t) dt}{X^\pm(t)(1 - K(t))(t+i)^m} = 0, \quad m = 1, 2, \dots, \varkappa, \quad (2)$$

is necessary and sufficient for problem (H) to have a solution; moreover, it is unique. The solution has the form (1), where one must take  $P_{\varkappa-1}(t) \equiv 0$ . Consequently, also in this case  $\Psi^\pm(t) \in R_0^\pm$  and is a solution of problem  $(\tilde{A}_b)$ .

Further, the explicit form of the solution of equation (A) is obtained from (1) by the inversion formula ((7), p. 57).

The solvability condition (2) for equation (A) can be rewritten in the form

$$\int_0^\infty \mu_m(x) f(x) dx = 0, \quad m = 1, 2, \dots, -\varkappa,$$

where  $\mu_1(x), \dots, \mu_{-\varkappa}(x)$  are linearly independent solutions of the equation

$$\mu(x) - \int_0^\infty k(y-x)\mu(y) dy = 0,$$

which is adjoint to  $(A_0)$ .

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