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Abstract

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ON THE THEORY OF TOPOLOGICAL RINGS

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By a **topological ring**, as usual, one means an associative ring which is a Hausdorff space and in which the ring operations are continuous. By an **ideal of a topological ring** we shall mean a two-sided, not necessarily closed, ideal. Recall that an ideal I of a topological ring R is called **topologically nilpotent** if for every neighborhood of zero V of the ring R there exists a natural number n such that

$$I^n \subseteq V.$$

Let R be an arbitrary topological ring. Denote by $\mathfrak{N}(R)$ the closure of the sum of all topologically nilpotent ideals of the ring R. For each ordinal number α we define a closed ideal $\mathfrak{N}_\alpha(R)$ as follows:

1) $\mathfrak{N}_0(R) = 0$.
2) Suppose that $\mathfrak{N}_\alpha(R)$ has been defined for every $\alpha < \beta$. If β is a limit ordinal, then put

$$\mathfrak{N}_\beta(R) = \left[\sum_{\alpha<\beta} \mathfrak{N}_\alpha(R)\right]_R * .$$

If, however, $\beta = \alpha + 1$, then for $\mathfrak{N}_\beta(R)$ we take the inverse image in R of the ideal $\mathfrak{N}(\overline{R})$ of the ring $\overline{R} = R/\mathfrak{N}_\alpha(R)$. There exists an ordinal τ such that

$$\mathfrak{N}_\tau(R) = \mathfrak{N}_{\tau+1}(R).$$

We shall call

$$L(R) = \mathfrak{N}_\tau(R)$$

the **topological Baer radical**.

From the construction of $L(R)$ it is clear that $R/L(R)$ contains no topologically nilpotent ideals.

We shall call a ring R **L-radical** if $L(R) = R$. Finally, we shall call a ring R **L-semisimple** if $L(R) = 0$, i.e., if R contains no topologically nilpotent ideals.

We have proved that:

1) In an arbitrary topological ring R, $L(R)$ is the intersection of all such closed ideals N of the ring R for which the quotient rings R/N contain no nonzero topologically nilpotent ideals.
2) For any ideal I of a topological ring R one has

$$L(I) \subseteq I \cap L(R).$$

It is known that, in the discrete case, for any ideal I of a ring R,

$$L(I) = I \cap L(R).$$

This equality also holds for a topological ring R possessing a complete system of group neighborhoods of zero (see (^{6})). However, as follows from Theorems 2 and 3 of the present work, there exists a topological ring $\widehat{R}$ and an ideal R' in $\widehat{R}$ such that

$$L(R') = 0 \subset R' \cap L(\widehat{R}) = R'.$$

In (^{6}) it is proved that if R is an everywhere dense subring of a topological ring $\widehat{R}$, then

$$L(R) \subseteq R \cap L(\widehat{R}).$$

From Theorems 4 and 5 of the present work it follows that there exist rings $\widehat{R}$ and $R' + R''$ such that $R' + R''$ is an everywhere dense subring of the ring $\widehat{R}$, and

$$L(R' + R'') = 0 \subset R' + R'' \cap L(\widehat{R}) = R'.$$

* If A is some subset of the topological space R, then by $[A]_R$ we shall denote the closure of A in R.

It is known that for a discrete ring $L(R)$ is a nil-ideal, and in a commutative discrete ring $L(R)$ coincides with the set of all nilpotent elements of the ring R.

In [6] it is shown that in a commutative bounded$*$ ring $L(R)$ coincides with the set of all topologically nilpotent elements. One can show that if a topological ring R has a full system of ideal neighborhoods of zero, then all elements of $L(R)$ are topologically nilpotent.

However, not in every topological ring R are the elements of $L(R)$ topologically nilpotent, even if R is a commutative ring. Thus, from Theorems 6 and 7 of the present paper it follows that there exists a commutative L-radical ring with identity.

Example 1. Let R' be the freely generated ring with a countable number of generators $x_1, x_2, \ldots$. The elements of the ring R' are all possible finite sums of words $x_{i_1} x_{i_2} \cdots x_{i_n}$. It is easy to verify that there are countably many such words. Consequently, all words can be numbered in a sequence $y_1, y_2, \ldots$. Every element $a \in R$ can be represented uniquely in the form $\sum_{i=1}^{\infty} \alpha_i y_i$, where the α_i are integers and only a finite number of $\alpha_i \neq 0$.

We shall say that the element $a' = \sum_{i=1}^{\infty} \alpha_i' y_i$ is smaller than the element $a = \sum_{i=1}^{\infty} \alpha_i y_i$, and shall write $a' < a$, if $\alpha_i = 0$ implies $\alpha_i' = 0$ and there exists an i_0 such that $\alpha_{i_0}' = 0$, but $\alpha_{i_0} \neq 0$. To each element $a \in R'$ we associate the number $s(a)$, equal to the degree of a regarded as a polynomial in the x_i, and to each element $0 \neq a \in R'$ we associate the number

$$m(a) = \min\{s(a') \mid 0 \neq a' \leqslant a\}.$$

Clearly, $m(a)$ is the least degree of the words y_i for which $\alpha_i \neq 0$. We shall put $m(0) = 0$.

To any sequence of integers $\alpha_1, \alpha_2, \ldots$, in which only a finite number of $\alpha_i \neq 0$, and to any natural number n, we associate the set $A^{(n)}_{\alpha_1, \alpha_2, \ldots}$ of all elements $a \in R$ which can be represented in the form

$$\sum_{j=1}^{k} a_j b_j + \sum_{i=1}^{\infty} \alpha_i y_i,$$

such that the following conditions are satisfied: a) for any $a_j' \leqslant a_j$, $b_j' \leqslant b_j$, $j = 1, 2, \ldots, k$,

$$\max\{s(a_1' b_1'), s(a_2' b_2'), \ldots, s(a_k' b_k')\} \geqslant pn,$$

where p is the number of the nonzero $a_j' b_j'$; b)

$$m\left(\sum_{i=1}^{\infty} \alpha_i y_i\right) \geqslant n \min(1, \sum |\alpha_i|).$$

Let Q be the discrete direct sum of fields of real numbers. Then Q is a regular ring**. Denote by R'' the ring of all infinite matrices over Q in which only finitely many elements are different from zero. Since the ring of all matrices of order n over a regular ring is a regular ring, and every element of R'' may be regarded as a matrix of order n, it follows that R'' is a regular ring. From the fact that every ideal of a regular ring is a regular ideal, it easily follows that R'' is L-semisimple in any topology.

For an arbitrary natural number n and an arbitrary sequence of integers $\alpha_1, \alpha_2, \ldots$, in which only finitely many $\alpha_i \neq 0$, define the set $B^{(n)}_{\alpha_1, \alpha_2, \ldots}$ of all matrices $\|\mathfrak{A}_{ij}\|$ for which

$$\sum_{i \geq n} \mathfrak{A}_{2i+1,2i} = \alpha_1, \alpha_2, \ldots,$$

and the remaining $\mathfrak{A}_{ij} = 0$.

Consider the ring with zero multiplication R''' over the discrete direct sum of groups of integers. The elements of the ring R''' are all possible sequences of integers $\mathfrak{A} = \alpha_1, \alpha_2, \ldots$, in which only a finite number of $\alpha_i \neq 0$.

Let $\hat{R}$ be the ring equal to the direct sum of the rings R', R'', R'''. The elements of the ring $\hat{R}$ are all possible triples of elements $(a, \|\mathfrak{A}_{ij}\|, \mathfrak{A})$, where $a \in R'$, $\|\mathfrak{A}_{ij}\| \in R''$, $\mathfrak{A} \in R'''$, and the operations are performed componentwise.

Denote by V_n the set of all elements $(a, \|\mathfrak{A}_{ij}\|, \mathfrak{A})$ for which $a \in A^{(n)}_{\alpha_1, \alpha_2, \ldots}$, $\|\mathfrak{A}_{ij}\| \in B^{(n)}_{\beta_1, \beta_2, \ldots}$, and $\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots = \mathfrak{A}$.

Theorem 1. The collection $\{V_n\}$ may be taken as a full system of neighborhoods of zero, so as to make $\hat{R}$ into a topological ring.

Theorem 2. In the topological ring $\hat{R}$, $L(\hat{R}) = R' + R'''$.

Proof. It can be shown that $R' + R'''$ is a closed ideal in $\hat{R}$. Since $\hat{R}/R' + R'''$ is algebraically isomorphic to R'', it follows that $\hat{R}/R' + R'''$ is L-semisimple. Hence $L(\hat{R}) \subseteq R' + R'''$. Conversely, it is obvious that $R''' \subseteq L(\hat{R})$. It can be shown that for every n one has

$$(R')^n \subseteq R''' + V_n \subseteq L(\hat{R}) + V_n.$$

Since $\hat{R}/L(\hat{R})$ contains no nonzero topologically nilpotent ideals, $R' + R''' = L(\hat{R})$.

Theorem 3. The ideal R', considered as a subring of the topological ring $\hat{R}$, is an L-semisimple ring.

$*$ A ring R is called **bounded** if for every neighborhood of zero V there exists a neighborhood of zero U such that $U \cdot R = \{ab \mid a \in U,\ b \in R\} \subseteq V$ and $R \cdot U \subseteq V$.

$**$ A ring R is called **regular** if the equation $x\xi x - x = 0$ is solvable in R for arbitrary x.

Proof. From the construction of V_n it follows easily that

$$V_n \cap R' = A^{(n)}_{0,0,\ldots}.$$

Consequently, the collection $\{A^{(n)}_{0,0,\ldots}\}$ is a full system of neighborhoods of zero in R'. For each element $a \in R'$ and each natural number r we construct an element $e_r \in (a)^r_{R'}$ such that $e_r \notin A^{(1)}_{0,0,\ldots}$.

Let n be a number such that a can be obtained from the generators $x_1, x_2, \ldots, x_n$. For the number r the number of words of degree not exceeding $rs(a)$ obtained with the help of the generators $x_1, x_2, \ldots, x_n$ is finite. Let these words be

$$y_{i_1}, y_{i_2}, \ldots, y_{i_l}.$$

It is easy to verify that

$$a^r = \sum_{j=1}^{l} \alpha_{i_j} y_{i_j}.$$

The ideal $(a)^r_{R'}$ contains the element

$$e_r = \sum_{i=1}^{t} x_i a^r x_i, \quad \text{where } t = (2rs(a)+3)(l+1).$$

It can be shown that e_r cannot be represented in the form

$$\sum_{i=1}^{p} a_i b_i,$$

so that condition a) for $n = 1$ is satisfied. From the arbitrariness of a and r it follows that R' contains no topologically nilpotent ideals.

Theorem 4. $R' + R''$ is an everywhere dense ideal of the topological ring $\hat{R}$.

Theorem 5. $R' + R''$ is an L-semisimple ring.

Proof. According to (2) and Theorem 2,

$$L(R' + R'') \subseteq (R' + R'') \cap L(\hat{R}) = (R' + R'') \cap (R' + R''') = R'.$$

Now let I be a topologically nilpotent ideal in $R' + R''$. Then

$$I \subseteq L(R' + R'') = R'.$$

Therefore I will be a topologically nilpotent ideal in R'. But this contradicts Theorem 3. Consequently, $R' + R''$ is an L-semisimple ring.

Example 2. Let P be the ring of all polynomials in a countable number of mutually commuting variables $x_1, x_2, \ldots$ with integer coefficients. Then P is a commutative ring with identity 1. To each element $a \in P$ assign the nonnegative number

$$l(a) = \max\{i \mid a \text{ depends on } x_i\}.$$

If a does not depend on any x_i, then we put $l(a) = 0$. For each natural number n define the set

$$P_n = \{b \in P \mid l(b) < n\}.$$

It is obvious that the P_n are subrings containing the subring of integers of the ring P.

Put

$$V'_n = \sum_k \sum_{i=1}^{k} P_{n,k}(1 - x_{n,k})^i,$$

where the sum is understood as the sum of the sets $P_{n,k}(1 - x_{n,k})^i$ in the ring P. For each number n define the ideal

$$V''_n = \sum_i P x_i^{n \cdot i}.$$

Denote $V'_n + V''_n$ by V_n.

Theorem 6. *The collection $\{V_n\}$ may be taken as a complete system of neighborhoods of zero in order to make P into a topological ring.*

Theorem 7. *The topological ring P is L-radical.*

Proof. Since for any neighborhood of zero V_n and any element $x_i \in P$ we have

$$(x_i)_P^{n \cdot i} = P x_i^{n \cdot i} \subseteq V_n,$$

all $x_i \in L(P)$. We now show that $1 \in L(P)$. Let V_n be an arbitrary neighborhood of zero. One can verify that $1 + V_n \ni x_n$. From the closedness of $L(P)$ it follows that $1 \in L(P)$. Since $L(P)$ is an ideal and contains the identity of the ring P, we have $L(P) = P$.

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Note: Figure translations are in progress. See original paper for figures.