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Abstract

Full Text

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TWO PARTICULAR SOLUTIONS OF THE PROBLEM OF THE MOTION OF A BODY HAVING A FIXED POINT

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Suppose that the center of gravity of the system lies on a principal axis of the gyration ellipsoid for the fixed point, and that along the same axis there is directed the vector $\vec{\lambda}$, characterizing internal cyclic motions (for example, the motion of a fluid circulating in multiply connected cavities of the body, or the motion of a flywheel of negligible weight).

The equations of motion of such a body ([1]), in the usual notation ([2]), have the form

$$A\frac{dp}{dt} = (B-C)qr,$$

$$B\frac{dq}{dt} = (C-A)rp - \lambda r - \gamma_3, \qquad C\frac{dr}{dt} = (A-B)pq + \lambda q + \gamma_2, \tag{1}$$

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \qquad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \qquad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2.$$

The known integrals are

$${}^1/_2\,(Ap^2 + Bq^2 + Cr^2) - \gamma_1 = E, \qquad (Ap+\lambda)\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = k, \tag{2}$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \Gamma^2.$$

In the solutions indicated below, $q, r, \gamma_1, \gamma_2, \gamma_3$ are expressed as functions of p. Substituting $q = q(p)$, $r = r(p)$ into (1), we determine, by quadrature, the dependence of p on t.

First solution.

$$\frac{C-B}{A}q^2 = \frac{A-C}{2B-A}p^2 + 2\lambda\,\frac{3BC-AB-C^2}{(2B-A)^2(2C-A)}\,p +$$

$$+\frac{2C-A}{(A-B)(A-C)}\left\{H - \lambda^2\frac{3BC-AB-C^2}{(2B-A)^3(2C-A)^3}\times\right.$$

$$\left.\times\left[A^3 - 2A^2(2C+B) + AC(3C+8B) - BC(5C+B)\right]\right\},$$

$$\frac{B-C}{A}r^2 = \frac{A-B}{2C-A}p^2 + 2\lambda\,\frac{3BC-AC-B^2}{(2C-A)^2(2B-A)}\,p + \frac{2B-A}{(A-C)(A-B)}\times \quad (3)$$

$$\times\left\{H - \lambda^2\frac{3BC-AC-B^2}{(2C-A)^3(2B-A)^3}\left[A^3 - 2A^2(2B+C) + AB(3+8C) - BC(5B+C)\right]\right\},$$

$$\gamma_1 = A\frac{(A-B)(A-C)}{(2B-A)(2C-A)}p^2 - \lambda A\frac{A^2(B+C) - 6ABC + 2BC(B+C)}{(2B-A)^2(2C-A)^2}p + H,$$

$$\gamma_2 = q\left\{\frac{(A-B)(A-C)}{2C-A}p + \lambda\left[C\frac{3BC-AC-B^2}{(2C-A)^2(2B-A)} - 1\right]\right\},$$

$$\gamma_3 = r\left\{\frac{(A-C)(A-B)}{2B-A}p + \lambda\left[B\frac{3BC-AB-C^2}{(2B-A)^2(2C-A)} - 1\right]\right\}.$$

Since q^2 and r^2 are polynomials of the second degree in p, p is an elliptic function of time.

This solution contains 7 independent parameters: $A, B, C, \lambda, H, p_0, \psi_0$ (ψ is the angle of precession).

On substituting (3) into (2), the constants E, k, and Γ are determined. Taking the quantity Γ as prescribed, we obtain two values for the parameter H:

$$H = \pm\sqrt{\Gamma^2 + \lambda^4\frac{A(B-C)^2(B+C-2A)\left[2A^2 - 3A(B+C) + 4BC\right]^2}{4(A-B)^2(A-C)^2(2B-A)^4(2C-A)^4}}$$

$$+\frac{\lambda^2 A}{(2B-A)(2C-A)}\left\{\frac{2A^4BC + A^3(B+C)(B^2-8BC+C^2) + 26A^2B^2C^2}{2(A-B)(A-C)(2B-A)^2(2C-A)^2}\right.$$

$$\left.+\frac{-12AB^2C^2(B+C) + 2B^2C^2(B+C)^2}{2(A-B)(A-C)(2B-A)^2(2C-A)^2} - 1\right\}.$$

In particular, for $\lambda = 0$, the indicated solution yields Steklov' s solution (3).

Second solution.

$$\frac{(10B-9C)^2}{36C^2}\,q^2+\left[p-\frac{\lambda}{2}\frac{10B-9C}{BC}-\frac{3}{2}\frac{(3C-2B)^2(3C-4B)}{bB(10B-9C)}\right]^2=\frac{3}{2}\frac{\lambda}{b}\frac{(3C-2B)(B-C)(10B-9C)}{B^2C}+\frac{9}{4}\frac{(3C-2B)^3(3C-4B)}{b^2B(10B-9C)^2}(18C^2-36BC+17B^2),$$

$$\begin{aligned}r^2={}&\frac{36b}{10B-9C}\,p^3-18\left[3\frac{b\lambda}{BC}+\frac{(3C-2B)(81C^2-156BC+61C^2)}{B(10B-9C)^2}\right]p^2\\&+9\left[3b\lambda^2\frac{10B-9C}{B^2C^2}+\frac{\lambda}{2B^2C(10B-9C)}(729C^3-1917BC^2+1520B^2C-388B^3)\right.\\&\qquad\left.+\frac{3}{2}\frac{(3C-2B)^2(3C-4B)}{bB^2(10B-9C)^3}(243C^3-648BC^2+495B^2C-122B^3)\right]p\\&-\frac{9}{2}b\lambda^3\frac{(10B-9C)^2}{B^3C^3}-\frac{3\lambda^2}{4B^3C^2(3C-2B)}(2187C^4-7533BC^3-9234B^2C^2\\&\qquad-5036B^3C+1064B^4)\\&+9\frac{\lambda}{b}\frac{3C-2B}{B^2C(10B-9C)^2}(729C^4-2754BC^3+3951B^2C+632B^4)\\&-\frac{27}{4}\frac{(B-C)(3C-2B)^3(3C-4B)}{b^2B^3(10B-9C)^4}(2187C^4-5832BC^3+4131B^2C^2\\&\qquad-30B^3C-488B^4),\end{aligned}$$

$$\begin{aligned}\gamma_1={}&\frac{18bC}{10B-9C}\,p^3-27\left[\frac{b\lambda}{B}+\frac{C(3C-2B)(27C^2-53BC+22B^2)}{B(10B-9C)^2}\right]p^2\\&+\left[\frac{27}{2}b\lambda^2\frac{10B-9C}{B^2C}+\frac{9\lambda}{4B^2(10B-9C)}(729C^3-1917BC^2+1528B^2C-388B^3)\right.\\&\qquad\left.+\frac{27}{4}\frac{C(3C-2B)^2(3C-4B)}{bB^2(10B-9C)^3}(243C^2-648BC^2+503B^2C-122B^3)\right]p\\&-\frac{9}{4}b\lambda^3\frac{(10B-9C)^2}{B^3C^2}-\frac{\lambda^2}{8B^3C}(2187C^3-5589BC^2+4104B^2C-916B^3)\\&+\frac{9}{8}\frac{\lambda}{b}\frac{(3C-2B)^2}{B^2(10B-9C)^2}(-243C^3+1161BC^2-1526B^2C+568B^3)\\&+\frac{9}{8}\frac{C(3C-2B)^4(3C-4B)}{b^2B^3(10B-9C)^4}(2187C^4-8748BC^3+12879B^2C^2\\&\qquad-8238B^3C+1888B^4),\end{aligned}$$

$$\gamma_2 = q\left\{3bp^2 - 3\frac{10B-9C}{BC}\left[b\lambda + \frac{C(3C-2B)(27C^2-54BC+22B^2)}{(10B-9C)^2}\right]p\right.$$
$$+\frac{3}{4}b\lambda^2\frac{10B-9C}{B^2C^2} + \frac{\lambda}{8B^2C}(729C^3-1917BC^2+1512B^2C-388B^3)$$
$$\left.+\frac{3}{8}\,\frac{(3C-2B)^2(3C-4B)}{bB^2(10B-9C)^2}(243C^3-648BC^2+495B^2C-122B^3)\right\},$$
$$\gamma_3 = r\left[3C\frac{3C-2B}{10B-9C}p - 2\lambda - 3\frac{C(3C-2B)^2(3C-4B)}{b(10B-9C)^2}\right].$$

This solution applies if the principal moments of inertia are subject to the condition

$$A = 18C\frac{B-C}{10B-9C}.$$

There are 6 independent parameters: $B, C, \lambda, b, p_0, \psi_0$.
The constants E, k, Γ are determined from the integrals (2).
For $\lambda = 0$, from the second solution there follows the solution of N. Kowalewski [4].

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Note: Figure translations are in progress. See original paper for figures.