ON THE ASYMPTOTIC TYPE OF PLANE-PARALLEL FLOW IN THE NEIGHBORHOOD OF THE CENTER OF A LAVAL NOZZLE

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Abstract

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ON THE ASYMPTOTIC TYPE OF PLANE-PARALLEL FLOW IN THE NEIGHBORHOOD OF THE CENTER OF A LAVAL NOZZLE

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To investigate the flow in the neighborhood of the center of a Laval nozzle, we shall use the equations describing gas flows in the transonic range of velocities ([1]):

$$-u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{1}$$

Here x and y are dimensionless Cartesian coordinates; the dimensionless functions u and v are related to the components of particle velocity U and V, the pressure p, density ρ, specific volume $\tau = 1/\rho$, specific entropy s, and speed of sound a by the relations

$$u = 2m_*\frac{U - a_*}{a_*}, \qquad v = 2m_*\frac{V}{a_*}, \qquad m_* = \frac{1}{2\rho_*^3 a_*^2}\left(\frac{\partial^2 p}{\partial \tau^2}\right)_s,$$

in which the asterisk denotes critical values of the gas parameters.

To determine the possible character of the flow in the neighborhood of the narrowest section of the channel, we pose for equations (1) the following Cauchy problem. Let, for $y = 0$, i.e., on the axis of symmetry of the flow,

$$u = -A_1|x|^k \quad \text{for } x < 0; \qquad u = A_2 x^k \quad \text{for } x > 0, \quad v = 0 \tag{2}$$

$$(A_1 > 0,\ A_2 > 0).$$

The two streamlines symmetric with respect to the x-axis in the flow constructed as a result of solving problem (2) will be taken as the walls of the nozzle.

The Cauchy problem under consideration with exponent $k = 1$ has been studied in a number of works ([2–11]); in this case the sonic curve is a parabola concave toward the oncoming flow. For $k = 2$ the transition line becomes a straight line perpendicular to the axis of symmetry of the nozzle ([12–15]); this shape of the transition line is preserved also for $k > 2$. In what follows we shall assume that the values of the exponent k lie in the interval $1 < k < 2$; the corresponding motions of the gas possess many features of flows with $k = 1$.

It is easy to show that the desired solution of the Cauchy problem is self-similar:

$$u = y^{2(n-1)} f(\xi), \qquad v = y^{3(n-1)} g(\xi), \qquad \xi = x/y^n, \qquad n = 2/(2-k). \tag{3}$$

Substitution of the formulas written above into equations (1) gives

$$f\frac{df}{d\xi} + n\xi\frac{dg}{d\xi} = 3(n-1)g, \qquad n\xi\frac{df}{d\xi} + \frac{dg}{d\xi} = 2(n-1)f.$$

Eliminating the function g from these, we obtain for f the second-order differential equation

$$(f - n^2\xi^2)\frac{d^2 f}{d\xi^2} + \left(\frac{df}{d\xi}\right)^2 + n(3n-5)\xi\frac{df}{d\xi} - 2(2n-3)(n-1)f = 0. \tag{4}$$

To simplify the qualitative study of problem (2), we set [9,10]

$$f = \xi^2 F(\eta), \qquad dF/d\eta = \Psi, \qquad \eta = \ln|\xi|. \tag{5}$$

In the new variables, equation (4) takes the form

$$\frac{d\Psi}{dF} = \frac{-6F - 5n\Psi + 6F^2 + 7F\Psi + \Psi^2}{(n^2 - F)\Psi}. \tag{6}$$

Thus, the problem of constructing the flow field in a neighborhood of the throat of a Laval nozzle has been reduced to determining the integrals of the ordinary differential equation (6), subject to conditions that are easily derived from the equalities (2). We shall not, however, consider solutions of the posed problem for arbitrary values of the constants A_1 and A_2 and of the exponent k. Our aim is to construct only those flows which have no singularities in the derivatives of the velocity components with respect to the coordinates on the C_-^0-characteristic arriving at the center of the nozzle. In them, the streamlines at the points of intersection with the C_-^0-characteristic likewise have no singularities; therefore, in order to obtain flows analytic in a neighborhood of the C_-^0-characteristic, it is not necessary to resort to any artificial devices by giving the nozzle walls a special shape. One such flow is well known; it is realized as a result of solving problem (2) with $k = 1$ and $A_1 = A_2$ [2–11]. Indeed, in this case the analytic character of the initial data also ensures the analytic character of the entire motion of the gas [16]. For $1 < k < 2$, the initial data (2) are not analytic; therefore the question of the absence of any singularities in the flow in a neighborhood of the C_-^0-characteristic cannot be decided on the basis of the Cauchy-Kovalevskaya theorem. To clarify it, we shall use the phase plane $F\Psi$, which gives the picture of the field of integral curves of equation (6).

The motion of the gas in the inlet part of the nozzle, between the x-axis and the C_-^0-characteristic arriving at its center, is represented by one of the integral curves of equation (6) connecting the singular points $A(0,0)$ and $C(n^2, -n(n+1))$, with an initial segment situated to the left of the Ψ-axis. The point A corresponds to the x-axis; passage through the point C signifies the intersection of the C_-^0-characteristic in the physical plane. By successive differentiation of formulas (5) it is easy to show that a discontinuity in the i-th derivative of the function $\Psi(F)$ corresponds to discontinuities in the $(i+1)$-st derivatives of the components of the velocity vector with respect to the coordinates. Therefore, the character of the singularity of the flow on the C_-^0-characteristic is determined by the expansion of the function $\Psi(F)$ in a neighborhood of the point C:

$$\Psi = -n^2 - n + a_1 \Delta F + a_2 (\Delta F)^2 + \cdots + b_1 (\Delta F)^\lambda + \cdots; \qquad \Delta F = F - n^2. \quad (7)$$

Here the coefficients a_i depend only on n, the constant b_1 is arbitrary, and the exponent λ of the first term of the irregular part is given by the formula

$$\lambda = \frac{5n-7}{n+1}. \quad (8)$$

As is known [17], the exponent λ is equal to the ratio of the roots of the characteristic equation determining the type of the singular point C. As long as its value is not equal to a positive integer, only one of the integrals (7) is holomorphic; its exact expression is [18]

$$\Psi = (n^2 - F)^{-1} \left[2F(F-n) - 2(n-1)F^{3/2} \right]. \quad (9)$$

The integral (9) is unsuitable for constructing the gas flow in the inlet part of the nozzle, since in this flow, as $y \to 0$, the vertical component of the velocity vector does not vanish.

Conversely, when λ is equal to a positive integer, analysis of equation (6) in accordance with the theorem of Briot and Bouquet ([17]) shows that, in a neighborhood of the point C, all its integrals will be holomorphic. When $k = 1$, then $n = 2$ and $\lambda = 1$ ([2–11]). For $1 < k < 2$ the values of n lie in the interval $2 < n < \infty$, and for $k = 4/3,\ 8/5,\ 20/11$ we have respectively $n = 3, 5, 11$ and $\lambda = 2, 3, 4$. As $k \to 2$, $n \to \infty$, and the exponent $\lambda \to 5$ ([12–15]). Let us apply the solutions of problem (2) for $k = 4/3,\ 8/5,\ 20/11$ to constructing a flow through a "natural" nozzle, in which singularities are absent in the derivatives of the velocity vector on the C_-^0-characteristic. For this purpose, as the flow depicted in the region beyond the C_-^0-characteristic, we choose an integral curve of equation (6), passing through the singular point C and serving as an analytic continuation of the segment onto which the gas motion in the inlet part of the channel is mapped. For the further investigation it is also convenient to use S. A. Chaplygin' s hodograph method ([19]), which makes it possible to pass from the nonlinear system of equations (1) to Tricomi' s linear equation for the function $y(u, v)$

$$\frac{\partial^2 y}{\partial u^2} - u\frac{\partial^2 y}{\partial v^2} = 0. \tag{10}$$

The solution of equation (10) satisfying the initial data (2) in the subsonic part of the flow, in a neighborhood of the axis $v = 0$, can be written in the form ([20])

$$y = -4^{1/2-j}(kA_1^{1/k})^{-1}v(9v^2 - 4u^3)^{j-1/2}F\left(\frac{1}{2} - j, \frac{2}{3} + j, \frac{3}{2}; \frac{9v^2}{9v^2 - 4u^3}\right), \tag{11}$$

$$j = (2-k)/(6k),$$

where F denotes, as usual, the hypergeometric function. In the supersonic part of the flow, in a neighborhood of the axis of symmetry of the flow, the solution of the Tricomi equation can be represented in the form ([20])

$$y = (kA_2^{1/k})^{-1}vu^{3j-3/2}F\left(\frac{1}{2} - j, \frac{5}{6} - j, \frac{3}{2}; \frac{9v^2}{4u^3}\right). \tag{12}$$

The form of the solution in a neighborhood of the transition line $u = 0$ and of the characteristics $v = \mp\frac{2}{3}u^{3/2}$ passing through the center of the nozzle is obtained from equalities (11) and (12) with the aid of formulas for analytic continuation of hypergeometric functions.

The results of the investigation of flows free of singularities in a neighborhood of the C_-^0-characteristic are as follows. For $k = 4/3$ the flow remains continuous throughout the whole region, and the C_+^0-characteristic issuing from the center of the nozzle carries discontinuities of third derivatives of the components of the velocity vector with respect to the coordinates. In this case, according to computations based on the hodograph method, the constant $A_2 = 0.415A_1$.

For $k = 8/5$ the flow proves impossible to continue beyond the C_-^0-characteristic because of the appearance in it of a limiting line arriving at the center of the nozzle.

For $k = 20/11$ a shock wave is formed in the flow; it originates at the center of the channel and then is carried downstream. In this case the solution of the system of equations (1), still having the form (3), must satisfy additional boundary conditions at the wave front: the equation of the shock polar ([20])

$$2(v_2 - v_3)^2 = (u_2 - u_3)^2(u_2 - u_3) \tag{13}$$

and the relation ([20])

$$u_2 \frac{dx_2}{dy} + v_2 = u_3 \frac{dx_2}{dy} + v_3, \tag{14}$$

which follows from the condition of continuity of the tangential component

velocity vector. In equalities (13) and (14) the indices refer to quantities on different sides of the shock front, and $x_2 = x_2(y)$ is the equation specifying its position. It is easy to show that this equation must have the form $\xi = \xi_2 = \text{const}$; hence, in the $F\Psi$-plane, conditions (13) and (14) are written in the form

$$F_2 + F_3 = 2n^2, \qquad \Psi_2 + \Psi_3 = -2n(7n - 5). \tag{15}$$

Calculations show that for $k = {}^{20}/_{11}$ $(n = 11)$ one can construct a solution of equation (6) satisfying the last two relations, which gives a gas flow analytic in a neighborhood of the C_0^0-characteristic. Such a flow is obtained for $A_2 = 0.111A_1$, and the velocity downstream of the compression shock remains supersonic.

It follows from the above that the flows with $k = {}^4/_3$ and $k = {}^{20}/_{11}$, as well as the previously known flow with $k = 1$, may be regarded as possessing an asymptotic character in a neighborhood of the center of the nozzle, but being realized for other wall shapes. The singularities present in them originate not on the walls, but in the flow itself, at the point of intersection of the sonic line with the axis of symmetry, and are then carried toward the exhaust part of the channel.

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Note: Figure translations are in progress. See original paper for figures.