ON THE APPLICATION OF THE METHOD OF INTEGRAL OPERATORS TO THE STUDY OF THE STEADY STATE OF ELASTIC-HEREDITARY SYSTEMS

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Abstract

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ON THE APPLICATION OF THE METHOD OF INTEGRAL OPERATORS TO THE STUDY OF THE STEADY STATE OF ELASTIC-HEREDITARY SYSTEMS

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To establish the influence of the time factor on the limiting state (as $t \to \infty$) of complex elastic-hereditary systems (for example, in the case of inhomogeneity of a material with rheological properties varying continuously with the coordinate), it is necessary to solve Volterra-type integral equations with kernels in the form of a sum of a finite or infinite number of terms. As a rule, it is not possible to obtain the solution of such problems in closed form.

In the present paper a theorem is proved that makes it possible to find directly, in closed form, the solution of the problem for the steady state of a complex elastic-hereditary system, without first solving the integral equations mentioned above and for any form of hereditary kernels invariant with respect to a change of the initial time origin.

Theorem. If, in the integral equation

$$y(t) - \int_0^t k(t-\tau)y(\tau)\,d\tau = f(t), \tag{1}$$

which describes the behavior of a certain elastic-hereditary system,

$$\lim_{t\to\infty} f(t) = F, \qquad f(t) > 0, \qquad k(t-\tau) > 0, \tag{2}$$

$$R = \int_0^\infty k(\theta)\,d\theta < 1, \tag{3}$$

then its solution $y(t)$ will be asymptotically stable. Moreover,

$$\lim_{t\to\infty} y(t) = F(1-R)^{-1}. \tag{4}$$

Proof of sufficiency. Making in the integral of equation (1) the change of variable $\tau = t - \theta$ and using the mean-value theorem, we find

$$y(t) = f(t)\left[1 - \frac{y(c)}{y(t)}\int_0^t k(\theta)\,d\theta\right]^{-1} \qquad (0 < c < t). \tag{5}$$

As is known, when the kernel $k(t-\tau)$ has a weak singularity, the solution of the integral equation (1) is continuous. Therefore, for the stability of the solution (5) as $t \to \infty$, it is sufficient that

$$R < \lim_{t\to\infty} \frac{y(t)}{y[c(t)]} = N. \tag{6}$$

Since a quasistatic problem is being considered (inertial forces are not taken into account), under the fulfillment of conditions (2) we have

$$\lim_{t\to\infty} \Delta y(t) = 0, \tag{7}$$

where $\Delta y(t)$ is the difference between two neighboring extreme values of $y(t)$. Therefore, if $N < 1$, then the stability of the solution of the integral equa-

of (1) is ensured. Now replacing the right-hand side in inequality (6) by unity, we obtain a sufficient condition for stability as $t \to \infty$ of the solution of equation (1), coinciding with (3).

Proof of necessity. Applying the Laplace transform to (1), we find

$$y(p) = f(p)[1 - k(p)]^{-1}. \tag{8}$$

If the function $y(t)$ is bounded, then, as is known ([1]),

$$\lim_{p\to 0} py(p) = \lim_{t\to\infty} y(t). \tag{9}$$

When conditions (2) and (3) are fulfilled, it follows from (8) and (9) that

$$\lim_{t\to\infty} y(t) = F\left[1 - \lim_{p\to\infty} pk(p)/p\right]^{-1} = F(1-R)^{-1}. \tag{10}$$

If $R = 1$, then, obviously, $\lim_{t\to\infty} y(t) = \infty$. Let now $R > 1$. Then from (10) we obtain $\lim_{t\to\infty} y(t) < 0$, which contradicts, according to (6) and (2), the continuity of the solution. Therefore formula (10) is invalid for $R > 1$ and, consequently (1), $\lim_{t\to\infty} y(t) = \infty$.

Corollary. From the necessity of condition (3) for stability of the solution of integral equation (1) and from (6) it follows that, if $\lim_{t\to\infty} y(t) = \infty$, then $N = 1$. For bounded $y(t)$, the validity of the equality $N = 1$ follows from comparing formulas (4) and (5), if in the latter one passes to the limit as $t \to \infty$. Then

$$\lim_{t\to\infty} \int_0^t k(t-\tau)y(\tau)\,d\tau = \lim_{t\to\infty} y(t) \int_0^t k(\theta)\,d\theta. \tag{11}$$

We shall demonstrate the effectiveness of the theorem proved above as applied to the problem of bending of a circular inhomogeneous plate clamped along its edges and loaded by a uniformly distributed load.

If the elastic moduli are the same for all points of the plate located on planes parallel to the middle plane, and vary symmetrically through the thickness, the stiffness of the plate in the case of ideal elasticity of the material may be calculated by the formula ([2])

$$D = 2\int_0^{h/2} \frac{Ez^2}{1-\nu^2}\,dz, \tag{12}$$

where h is the thickness of the plate; z is the coordinate of a point, measured from the middle surface, $E = E(z) = E(-z)$ and $\nu = \nu(z) = \nu(-z)$.

The deflection of the plate at any point can then be calculated by the formula

$$W = A(r)q(t)D^{-1}, \tag{13}$$

where $q(t) > 0$, $\lim_{t\to\infty} q(t) = Q$.

In order to take into account the influence of the time factor on the deflection of the plate, whose material possesses hereditary properties varying symmetrically through the thickness, we use Volterra' s principle ([3]), according to which in (12) and (13) we substitute operators in place of the elastic constants:

$$E_t = E[1 - \varkappa \mathcal{E}_a^*(-\beta)], \qquad \nu_t = \nu[1 + \delta \mathcal{E}_a^*(-\beta)], \tag{14}$$

where $\delta = (1-2\nu)\varkappa/2\nu$, $\varkappa = \varkappa(z) = \varkappa(-z)$, $\beta = \beta(z) = \beta(-z)$, $\mathcal{E}_a^*(-\beta)y(t) =$

$$= \int_0^t \mathcal{E}_a(-\beta, t-\tau)y(\tau)\,d\tau,$$

$\mathcal{E}_a(-\beta, t-\tau)$ is a kernel of the Yu. N. Rabotnov type ([3]).

Then

$$D_t = D\left\{1 - \frac{1}{D}\int_0^{h/2} \left[a_1 \mathcal{E}_\alpha^*(-\beta_1) + a_2 \mathcal{E}_\alpha^*(-\beta_2)\right] dz\right\}, \tag{15}$$

where

$$a_1 = \frac{E\chi z^2}{(1-\nu)^2}, \qquad a_2 = \frac{3E\chi z^2}{(1+\nu)^2}, \qquad \beta_1 = \beta - \frac{\nu\delta}{1-\nu}, \qquad \beta_2 = \beta + \frac{\nu\delta}{1+\nu}.$$

Here the fundamental properties of integral operators with kernels due to Yu. N. Rabotnov ([3]) have been used. Substituting (15) into (13) and using (3) and (4), we find that, for

$$\int_0^{h/2} dz \int_0^\infty \left[a_1 \mathcal{E}_\alpha(-\beta_1, \theta) + a_2 \mathcal{E}_\alpha(-\beta_2, \theta)\right] d\theta < D \tag{16}$$

the steady deflection can be computed by the formula

$$\lim_{t\to\infty} W(t) = \frac{A(r)Q}{D}\left\{1 - \frac{1}{D}\int_0^{h/2} dz \int_0^\infty \left[a_1 \mathcal{E}_\alpha(-\beta_1, \theta) + a_2 \mathcal{E}_\alpha(-\beta_2, \theta)\right] d\theta\right\}^{-1}. \tag{17}$$

In computing the improper integral in (17), it is advisable to use the following approximation ([4]):

$$\mathcal{E}_\alpha^*(-\beta)\cdot 1 \approx \frac{1}{\beta}\left[1 - \exp\left(-\gamma\beta t^{1-\alpha}\right)\right], \qquad \gamma = (1-\alpha)^{1-\alpha}.$$

Then

$$\lim_{t\to\infty} W(t) = \frac{A(r)Q}{D}\left[1 - \frac{1}{D}\int_0^{h/2}\left(\frac{a_1}{\beta_1} + \frac{a_2}{\beta_2}\right) dz\right]^{-1}.$$

If this problem is solved directly, without using the theorem proved above, it is necessary to find the inverse of a function of operators whose parameters depend on the coordinate with respect to which integration is carried out. Methods for inverting such operators have not yet been established.

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Note: Figure translations are in progress. See original paper for figures.