Mathematics

G. Kh. Berman

1964

SovietRxiv

Abstract

Full Text

Mathematics

G. Kh. Berman

FUNCTORS IN THE CATEGORY OF LOCALLY CONVEX SPACES

(Presented by Academician P. S. Novikov on 23 IX 1963)

Spaces of sequences whose terms are elements of some locally convex space have been studied by many authors from various points of view. In the note ([1]), A. C. Schwartz considers such spaces as values of functors in the category of Banach spaces. In the present note, following the ideas of A. C. Schwartz ([1,2]), we shall consider spaces of sequences as values of functors in the category of locally convex spaces.

1°. Let λ be a perfect space of numerical sequences ([3,4]), and let λ^* be its dual. λ and λ^* are in duality ([5]) with respect to the bilinear form

$$\langle \vec{\xi}_n, \vec{a}_n \rangle = \sum_{n=1}^{\infty} a_n \xi_n \qquad (\vec{\xi}_n \in \lambda,\ \vec{a}_n \in \lambda^*).$$

$M \subset \lambda$ is called **bounded** ([3]) if for every element $\vec{a}_n \in \lambda^*$ there exists a positive number ρ such that

$$\sum_{n=1}^{\infty} |a_n \xi_n| \leq \rho$$

for all $\vec{\xi}_n \in M$. The **normal hull** of M is the totality of all $\vec{\eta}_n \in \lambda$ such that $|\eta_n| \leq |\xi_n|$ for some $\vec{\xi}_n \in M$. M is called **normal** ([3,4]) if it coincides with its normal hull. A normal hull of a bounded set is bounded. A system $\mathfrak{M}$ of sets from λ^* is called a **normal topologizing system for** λ ([3]) if the following conditions are satisfied: 1) every $M \in \mathfrak{M}$ is bounded; 2) if $M_1, M_2 \in \mathfrak{M}$, then there exists $M \in \mathfrak{M}$ such that $M_1 \cup M_2 \subset M$; 3) if $M \in \mathfrak{M}$ and $\rho > 0$, then $\rho M \in \mathfrak{M}$; 4) $\mathfrak{M}$ covers λ^*; 5) together with each of its sets $\mathfrak{M}$ also contains its normal hull. The polars in λ of the sets of the system $\mathfrak{M}$ form in λ a fundamental system of neighborhoods of zero for a separable locally convex $\mathfrak{M}$-topology ([3]). We shall denote by $\lambda_{\mathfrak{M}}$ the space λ endowed with this topology. Every $\mathfrak{M}$-bounded set is bounded ([3]), i.e. one may speak of bounded sets in λ without indicating the $\mathfrak{M}$-topology.

E will denote a separable locally convex space; $\mathfrak{U}$ the collection of all neighborhoods of zero U in E; p_U the seminorm corresponding to the neighborhood $U \in \mathfrak{U}$; $\lambda(E)$ the space of all sequences $x_n,\ x_n \in E$, such that $\overrightarrow{p_U(x_n)} \in \lambda$ for all $U \in \mathfrak{U}$. The sets

$$(M, U) = \left\{ x_n \in \lambda(E) : \sum_{n=1}^{\infty} |a_n|\, p_U(x_n) \leq 1 \text{ for } \vec{a}_n \in M \right\} \qquad (M \in \mathfrak{M},\ U \in \mathfrak{U})$$

form a fundamental system of neighborhoods of zero in $\lambda(E)$ for some separable locally convex topology. The space $\lambda(E)$, endowed with this topology, will be denoted by $\lambda_{\mathfrak{M}}(E)$. Everywhere below we shall assume that the normal topologizing system $\mathfrak{M}$ for λ consists of absolutely convex sets that are bicompact in the weak topology $\sigma(\lambda^*, \lambda)$.

Theorem 1. *If E is complete, then $\lambda_{\mathfrak{M}}(E)$ is complete.*

Let $\lambda \otimes E$ be the tensor product of the vector spaces λ and E.

If E is locally convex, $\lambda \otimes E$ is embedded one-to-one in $\lambda(E)$. We denote $\lambda \otimes E$ in the topology induced from $\lambda_{\mathfrak{M}}(E)$ by $\lambda \otimes_{\mathfrak{M}} E$, and its completion by $\lambda \hat{\otimes}_{\mathfrak{M}} E$.

Theorem 2. *If E is complete, then $\lambda_{\mathfrak{M}}(E)$ and $\lambda \hat{\otimes}_{\mathfrak{M}} E$ are isomorphic.*

Let $\lambda_{\mathfrak{M}} \hat{\otimes} E$ and $\lambda_{\mathfrak{M}} \hat{\otimes} E$ be the known [6] topological tensor products of the locally convex spaces E and $\lambda_{\mathfrak{M}}$.

Theorem 3. *The canonical mappings $\lambda_{\mathfrak{M}} \hat{\otimes} E$ into $\lambda \hat{\otimes}_{\mathfrak{M}} E$ and $\lambda \hat{\otimes}_{\mathfrak{M}} E$ into $\lambda_{\mathfrak{M}} \hat{\otimes} E$ are continuous.*

Theorem 4. *If E is a complete separated nuclear space, then the sets*

$$[M,\ U] = \left\{ x_n \in \lambda(E) : \sum_{n=1}^{\infty} \alpha_n x_n \in U \text{ for } \alpha_n \in M \right\}$$

$(M \in \mathfrak{M},\ U \in \mathfrak{U})$ form a fundamental system of neighborhoods of zero in $\lambda_{\mathfrak{M}}(E)$.

2°. $L(X, Y)$, where X and Y are locally convex spaces, will denote the space of all continuous linear mappings of X into Y, endowed with the topology of bounded convergence; it will be denoted by $[X \to Y]$. We shall consider only categories of separated locally convex spaces containing the one-dimensional space I and such that $\mathrm{Hom}(X, Y) = L(X, Y)$ for all spaces X, Y of the category. All functors under consideration $F : \mathcal{K} \to \mathcal{K}_1$ will be assumed linear, i.e. such that the mapping $\varphi \to F\varphi$ from the space $L(X, Y)$ into $L(FX, FY)$ is linear. A functor $F : \mathcal{K} \to \mathcal{K}_1$ will be called continuous if $\varphi \to F\varphi$ is a continuous mapping of $[X \to Y]$ into $[FX \to FY]$. The category of all separated locally convex spaces will be denoted by $\mathcal{L}$, that of all (F)-spaces by $\mathcal{F}$, and that of all nuclear (F)-spaces by $\mathcal{F}N$.

Let A be a locally convex space. The functor $\Omega_A : \mathcal{K} \to \mathcal{L}$ is defined as follows:

$$\Omega_A X = [A \to X], \qquad \Omega_A \varphi(a) = \varphi \cdot a$$

$(\varphi \in L(X,Y),\ a \in \Omega_A X)$. The functor Ω_A is continuous.

The functor $\Sigma_A : \mathcal{K} \to \mathcal{L}$ is defined as follows:

$$\Sigma_A X = A \hat{\otimes} X, \qquad \Sigma_A \varphi = 1_A \hat{\otimes} \varphi$$

[6] ($1_A : A \to A$ is the identity mapping). If $A \in \mathcal{F}$, then $\Sigma_A : \mathcal{F}N \to \mathcal{L}$ is continuous; if $A \in \mathcal{F}N$, then $\Sigma_A : \mathcal{F} \to \mathcal{L}$ and $\Sigma_A : \mathcal{F}N \to \mathcal{L}$ are continuous.

The functor $\Lambda_{\mathfrak{M}} : \mathcal{K} \to \mathcal{L}$, generated by the space $\lambda_{\mathfrak{M}}$, is defined as follows:

$$\Lambda_{\mathfrak{M}} X = \lambda_{\mathfrak{M}}(X), \qquad \Lambda_{\mathfrak{M}} \varphi(x_n) = \varphi x_n.$$

Theorem 5. *The functor $\Lambda_{\mathfrak{M}} : \mathcal{F}N \to \mathcal{L}$ is continuous.*

3°. By virtue of the linearity of all the objects and morphisms under consideration, vector operations are naturally defined on mappings of functors; therefore the collection of mappings of a functor F into a functor S, provided that it is a set, forms a vector space; we shall denote it by $H(F,S)$. The correspondence $\alpha \to \alpha_X$ establishes a canonical mapping of $H(F,S)$ into $L(FX,SX)$.

Lemma 1. *The canonical mapping $\alpha \to \alpha_I$ is an isomorphism of the spaces $H(\Sigma_A, \Sigma_B)$ and $L(A,B)$.*

Lemma 2. *If $F : \mathcal{K} \to \mathcal{L}$ is continuous and $A \in \mathcal{K}$, then $H(\Omega_A, F)$ is isomorphic to FA with respect to the mapping $\alpha \to \alpha_A(1_A)$.*

Let the functors $F, S : \mathcal{K} \to \mathcal{L}$ be such that the collection of mappings of F into S is a set. Denote by τ_X $(X \in \mathcal{K})$ the inverse image of the topology of the space $[FX \to SX]$ with respect to the canonical mapping $H(F,S)$ into $[FX \to SX]$. The collection of topologies τ_X on $H(F,S)$ forms a set; let τ be the upper bound of this set of topologies. The space $H(F,S)$ in the topology τ is a separated locally convex space, which we shall denote by $\{F \to S\}$. For example, if $A, B \in \mathcal{F}N$ and $\Sigma_A, \Sigma_B : \mathcal{F}N \to \mathcal{L}$, then $\{\Sigma_A \to \Sigma_B\}$ is isomorphic to $[A \to B]$.

The functor $F : \mathcal{K} \to \mathcal{L}$ is called **regular** if the vector isomorphism from Lemma 2 is a topological isomorphism.

Theorem 6. The functor $\Lambda_{\mathfrak{M}} : \mathcal{F}N \to \mathcal{L}$ is regular.

One can also prove the regularity of the functors $\Sigma_A, \Omega_A : \mathcal{F}N \to \mathcal{L}$ under the condition $A \in \mathcal{F}N$.

We shall call the functor $\overline{F} : \mathcal{K}_1 \to \mathcal{L}$ **dual** (1) to the functor $F : \mathcal{K} \to \mathcal{L}$ if, for every $A \in \mathcal{K}_1$, $\overline{F}A = \{F \to \Sigma_A\}$, and for every $\varphi \in L(A,B)$ $(A, B \in \mathcal{K}_1)$

one has $\overline{F}\varphi(a) = \tilde{\varphi} \circ a$, where $a \in \overline{F}A$, and $\tilde{\varphi} : \Sigma_A \to \Sigma_B$ is defined by the isomorphism from Lemma 1.

Theorem 7. For any functor $F : \mathcal{F}N \to \mathcal{L}$ there exists, and moreover is continuous, the dual functor $\overline{F} : \mathcal{F}N \to \mathcal{L}$.

One can prove that if $A \in \mathcal{F}N$ and $\Omega_A, \Sigma_A : \mathcal{F}N \to \mathcal{L}$, then $\overline{\Sigma}_A = \Omega_A$ and $\overline{\Omega}_A = \Sigma_A$ (cf. ([1])).

If $\lambda_{\mathfrak{N}}$ is semireflexive, then every closed bounded set in $\lambda_{\mathfrak{N}}$ is weakly bicompact ([5]). Since the weakly closed normal hull of a weakly bicompact set from λ is weakly bicompact ([4]), and the normal hull of an absolutely convex set from λ is absolutely convex, in the case of semireflexive $\lambda_{\mathfrak{N}}$ there exists a fundamental system $\mathfrak{N}$ of bounded sets from λ, consisting of normal absolutely convex weakly bicompact sets and being a normal topologizing system for λ^*. If by $\Lambda^*_{\mathfrak{N}}$ we denote the functor corresponding to the space $\lambda^*_{\mathfrak{N}}$, then the following holds.

Theorem 8. If $\lambda_{\mathfrak{N}}$ is semireflexive, then the functor $\Lambda^*_{\mathfrak{N}} : \mathcal{F}N \to \mathcal{L}$ is dual to the functor $\Lambda_{\mathfrak{N}} : \mathcal{F}N \to \mathcal{L}$.

4°. For every functor $F : \mathcal{F}N \to \mathcal{L}$ one can construct a bilinear mapping

$$\omega_{X,Y} : \overline{F}X \times FY \to X \widehat{\otimes}_\pi Y$$

$(X, Y \in \mathcal{F}N)$, defined by the relation

$$\omega_{X,Y}(\bar{x}, y) = \bar{x}_Y(y),$$

where $\bar{x} \in \overline{F}X,\ y \in FY$ ($\bar{x}_Y$ is the image of the element $\bar{x}$ under the canonical mapping $\overline{F}X = \{F \to \Sigma_X\}$ into $[FX \to \Sigma_{XY}]$). Let $\bar{b}_X$ (respectively b_Y) be the collection of all bounded sets of the space $\overline{F}X$ (respectively FY). If $\omega_{X,Y}(\bar{b}_X - -b_Y)$ is hyponecontinuous ([5]) for every pair $X, Y \in \mathcal{F}N$, then the correspondence

$$y \to \{\omega_{X,Y}(\cdot, y)\}_{X \in \mathcal{F}N}$$

defines, for each $Y \in \mathcal{F}N$, a mapping $\omega_Y \in L(FY, \overline{\overline{F}}Y)$ in such a way that $\omega = \{\omega_Y\}_{Y \in \mathcal{F}N}$ is a mapping of F into $\overline{\overline{F}}$, which we shall call **natural**.

The functor $F : \mathcal{F}N \to \mathcal{L}$ is called ([1]) **reflexive** if the natural mapping $\omega : F \to \overline{\overline{F}}$ is an isomorphism.

Theorem 9. The functor dual to any reflexive functor $F : \mathcal{F}N \to \mathcal{L}$ is also reflexive.

Denote by $\mathfrak{N}$ (respectively $\mathfrak{N}^*$) the system of all normal absolutely convex bounded sets from λ (respectively λ^*), closed in the weak topology $\sigma(\lambda, \lambda^*)$ (respectively $\sigma(\lambda^*, \lambda)$).

Theorem 10. If $\lambda_{\mathfrak{N}^*}$ is reflexive, then $\Lambda_{\mathfrak{N}^*} : \mathcal{F}N \to \mathcal{L}$ is reflexive, and moreover $\overline{\Lambda}_{\mathfrak{N}^*} = \Lambda_{\mathfrak{N}}$.

Hence, in particular, it follows that if by Λ^p we denote the functor generated by the space l^p $(p > 1)$, then

$$\overline{\Lambda^p} = \Lambda^q \left(\frac{1}{p} + \frac{1}{q} = 1\right),$$

and these functors are reflexive.

I take this opportunity to express my deep gratitude to A. S. Schwartz for very valuable conversations.

Received
20 IX 1963

CITED LITERATURE

[1] A. S. Schwartz, DAN, **149**, No. 1 (1963).
[2] A. S. Schwartz, DAN, **148**, No. 2 (1963).
[3] A. Pietsch, *Verallgemeinerte vollkommene Folgenräume*, Berlin, 1962.
[4] G. Köthe, *Topologische lineare Räume*, Berlin, 1960.
[5] N. Bourbaki, *Topological Vector Spaces*, IL, 1959.
[6] A. Grothendieck, Mem. Am. Math. Soc., No. 16 (1955).

Note: Figure translations are in progress. See original paper for figures.