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Abstract

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MATHEMATICS

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ON THE COMPLETENESS OF THE SYSTEM OF EIGEN AND ASSOCIATED ELEMENTS OF OPERATORS THAT ARE RATIONAL FUNCTIONS OF A PARAMETER

(Presented by Academician M. V. Keldysh, 15 VI 1964)

Let H be a completely continuous self-adjoint operator acting in a separable Hilbert space $\mathcal{H}$, and let A_i $(i = 1, 2, \dots, n-1)$ be completely continuous operators.

Consider the operator $L = L_1 + L_2 + A$, where

$$L_1 = \sum_{i=1}^{n-1} \lambda^i A_i H^i + \lambda^n H^n,$$

and A and L_2 are completely continuous operators, with L_2 depending on the complex parameter λ.

In the present paper we establish certain sufficient conditions for the multiple completeness of the system of eigen and associated (e.a.) elements of the operator L.

Definition. We shall say that a function $\Phi(\lambda)$ has finite order of growth at the point λ_0 if there exist numbers $R > 0$ and ρ, and such functions $D(\lambda)$ and $\Delta(\lambda)$, that $\Phi(\lambda) = D(\lambda)/\Delta(\lambda)$ and, for $|\lambda - \lambda_0| \leq R$, the inequalities

$$|D(\lambda)| \leq e^{|\lambda-\lambda_0|^{-\rho}}, \qquad |\Delta(\lambda)| \leq e^{|\lambda-\lambda_0|^{-\rho}}.$$

The number $\rho = \inf \rho'$ (the infimum being taken over all ρ' satisfying the last inequalities) will be called the order of growth of the function $\Phi(\lambda)$ at the point λ_0. If one can choose $a(\lambda)$ so that

$$|D(\lambda)| \le e^{a(\lambda)|\lambda-\lambda_0|^{-\rho}}, \qquad |\Delta(\lambda)| \le e^{a(\lambda)|\lambda-\lambda_0|^{-\rho}},$$

and $\lim_{|\lambda|\to|\lambda_0|} |a(\lambda)| = 0$ as $\lambda \to \lambda_0$, then the function $\Phi(\lambda)$ will be called a function of minimal type. In a corresponding way one may introduce the notions of functions of maximal and normal types.

If $\Phi(\lambda)$ is an operator function, then $D(\lambda)$ is also assumed to be an operator function, and in the corresponding inequality for $D(\lambda)$ the modulus is replaced by the norm.

The following theorem is valid (a theorem of Phragmén–Lindelöf type for a neighborhood of a point).

Theorem 1. Let $\Phi(\lambda)$ be an analytic function having an isolated singularity at the point λ_0 and satisfying the following conditions:
1) in a neighborhood of the point λ_0 the function $\Phi(\lambda)$ has finite order ρ;
2) there exists a system of rays issuing from the point λ_0 such that the angle α between neighboring rays of the system is less than π/ρ ($\alpha < \pi/\rho$), and on all rays of this system the function $\Phi(\lambda)$ is bounded in modulus.

Then the function $\Phi(\lambda)$ is bounded in modulus in some neighborhood of the point λ_0.

If $\Phi(\lambda)$ has minimal type, then the theorem remains valid also when the angle between neighboring rays is equal to π/ρ ($\alpha = \pi/\rho$).

Lemma 1. *If $M(\lambda)$ is a linear operator satisfying the condition $\|M(\lambda)\| \le M$ for all $|\lambda| \ge R \ge 0$, and the operator A has finite order ρ, then the resolvent of the operator $\lambda M(\lambda)A$ is an operator-valued function of minimal type, and the order of its growth in a neighborhood of the infinitely distant point does not exceed ρ.*

We briefly indicate the idea of the proof.

Consider the operator $\lambda M(\mu)A$. By a theorem of M. V. Keldysh (see ([2,3]))

$$R(\lambda, \mu) = [E - \lambda M(\mu)A]^{-1} = \frac{D_\mu(A)}{\Delta_\mu(A)},$$

where $D_\mu(A)$ and $\Delta_\mu(A)$ are functions of λ, the orders of growth of which do not exceed ρ. By simple calculations it is proved that the estimates of M. V. Keldysh for $D_\mu(\lambda)$ and $\Delta_\mu(\lambda)$ are uniform with respect to μ for $|\mu| \ge R$.

Therefore, putting $\mu = \lambda$ for $|\lambda| \ge R$, we obtain that the order of growth of the resolvent of the operator $\lambda M(\lambda)A$ in a neighborhood of the infinitely distant point does not exceed ρ. Similar arguments lead to the assertion of the lemma concerning the type of the resolvent $(E - \lambda M(\lambda)A)^{-1}$.

Lemma 2. *If* $\lim_{|\lambda|\to\infty}\|L_2(\lambda)\| = 0$ *and* H *is an operator of finite order* ρ, *then the order of growth of the resolvent of the operator* $L = L_1 + L_2 + A$ *in a neighborhood of the infinitely distant point does not exceed* ρ.

Proof. We have the inequality

$$\|(E - L_1 - A - L_2)^{-1}\| \leq \|(E-A)^{-1}\| \cdot \|(E-(E-A)^{-1}L_2)^{-1}\| \times$$

$$\times \|(E-(E-(E-A)^{-1}L_2)(E-A)^{-1}L_1)^{-1}\|.$$

Without loss of generality one may assume that the operator $(E-A)^{-1}$ exists and is bounded. Then the operator $(E-(E-A)^{-1}L_2)^{-1}$ will be bounded for sufficiently large $|\lambda|$, and therefore it suffices to investigate the operator

$$(E-(E-(E-A)^{-1}L_2)^{-1}(E-A)^{-1}L_1)^{-1}.$$

For this purpose consider the equation

$$y = \widetilde{M}(\lambda)L_1(y) + f,$$

where

$$\widetilde{M}(\lambda) = (E-(E-A)^{-1}L_2)^{-1}(E-A)^{-1}.$$

Rewrite this equation in the form of the system

$$y_0 = \lambda M(\lambda)Hy_1 + M(\lambda)\sum_{k=n-1}^{2} A_{kHy_{n-k+1}} + \lambda M(\lambda)A_1Hy_0 + f_0,$$

$$y_1 = \lambda H y_2 + f_1, \tag{1}$$

$$\dots\dots\dots\dots\dots\dots$$

$$y_{n-1} = \lambda H y_0 + f_{n-1}.$$

Following M. V. Keldysh, introduce the space $\mathfrak{H}^n$ in the following way: the elements of $\mathfrak{H}^n$ will be

$$\tilde{x} = \{x_0, x_1, \dots, x_{n-1}\},$$

where x_i $(i = 0, 1, \dots, n-1)$ are arbitrary ordered systems of n elements belonging to $\mathfrak{H}$. The scalar product $[\tilde{x}, \tilde{y}]$ of the elements

$$\tilde{x} = \{x_0, x_1, \dots, x_{n-1}\}, \qquad \tilde{y} = \{y_0, y_1, \dots, y_{n-1}\}$$

in $\mathfrak{H}^n$ is defined as follows:

$$[\tilde{x}, \tilde{y}] = \sum_{i=0}^{n-1} (x_i, y_i),$$

where (x_i, y_i) $(i = 0, 1, \dots, n-1)$ is the scalar product in $\mathfrak{H}$.

The system (1) in the space $\mathfrak{H}^n$ can be rewritten in the form

$$\tilde{y} = \lambda \widetilde{K} \widetilde{H} \tilde{y} + \tilde{f},$$

where $\widetilde{K}$ and $\widetilde{H}$ have the form

$$\widetilde{K} = \begin{pmatrix} M(\lambda)A_1 & M(\lambda) & M(\lambda)A_{n-1} & \cdots & M(\lambda)A_2 \\ & E & & & \\ & & \ddots & & \\ & & & E & \\ & & E & & \\ E & & & & \end{pmatrix}, \qquad \widetilde{H} = \begin{pmatrix} H & 0 & \cdots & 0 \\ 0 & H & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & H \end{pmatrix}.$$

It is not difficult to prove that $\widetilde{K}(\lambda)$ is bounded for all $|\lambda| > c$, where c is some number. $\widetilde{H}$ is an operator of order ρ; therefore, by Lemma 1, in a neighborhood of the infinitely distant point the resolvent $(\widetilde{E} - \lambda \widetilde{K}(\lambda) \widetilde{H})^{-1}$ has order of growth not exceeding ρ and minimal type. But since

$$\left\| (E - \sum \lambda^i M(\lambda) A_i H^i - \lambda^n M(\lambda) H^n)^{-1} \right\|_{\mathfrak{H}} \leq \left\| (\widetilde{E} - \lambda \widetilde{K}(\lambda) \widetilde{H})^{-1} \right\|_{\mathfrak{H}^n},$$

it is not difficult to prove the assertion of the lemma.

Lemma 3. *Under the conditions of Lemma 2, in a neighborhood of the infinitely distant point the resolvent of the operator $L(\lambda)$ can be represented in the form*

$$(E - L(\lambda))^{-1} = (E + B(\lambda))(E - \lambda^n H^n)^{-1},$$

where

$$\lim_{|\lambda| \to \infty} \|B(\lambda)\| = 0$$

uniformly with respect to the argument λ inside the angles

$$\frac{\pi k}{n} + \varepsilon \leq \arg \lambda \leq \frac{\pi(k+1)}{n} - \varepsilon \qquad (2)$$

for any $\varepsilon > 0$.

Proof. Consider the equation

$$y = (L_1(\lambda) + A + L_2(\lambda))y + f.$$

We have

$$(E - \lambda^n H^n)y = (L_1(\lambda) + A + L_2(\lambda) - \lambda^n H^n)y + f.$$

Hence we obtain

$$y = (E - \lambda^n H^n)^{-1}(L_1(\lambda) + A - \lambda^n H^n)y + (E - \lambda^n H^n)^{-1} L_2(\lambda)y +$$

$$+(E - \lambda^n H^n)^{-1} f.$$

Since $\lim_{|\lambda|\to\infty} \|L_2(\lambda)\| = 0$ and the operator $(E - \lambda^n H^n)^{-1}$ is bounded on the angles (2) uniformly with respect to the argument λ, it follows that

$$\|(E - \lambda^n H^n)^{-1} L_2\| \to 0$$

on the angles (2) as $|\lambda| \to \infty$, uniformly with respect to the argument λ. It is known (2) that as $|\lambda| \to \infty$

$$\|(E - \lambda^n H^n)^{-1}(L_1(\lambda) + A - \lambda^n H^n)\| \to 0$$

on the angles (2). Consequently, as $|\lambda| \to \infty$

$$\|(E - \lambda^n H^n)^{-1}(L(\lambda) - \lambda^n H^n)\| \to 0$$

inside the angles (2). Therefore, if we set

$$c(\lambda) = (E - \lambda^n H^n)^{-1}(L(\lambda) - \lambda^n H^n),$$

then for sufficiently large $|\lambda|$ the operator $(E - c(\lambda))^{-1}$ exists and is defined by the equality

$$(E - c(\lambda))^{-1} = \sum_{k=0}^{\infty} c^k(\lambda) = E + B(\lambda),$$

where

$$\|B(\lambda)\| \le \frac{\|c(\lambda)\|}{1 - \|c(\lambda)\|}$$

on the rays from (2). This proves the lemma.

Remark 1. The study of the behavior of the resolvent of an operator in a neighborhood of a finite point of the plane can, by means of a fractional-linear transformation, be reduced to the study of it in a neighborhood of the infinitely distant point. Therefore analogues of Lemmas 1, 2, 3, in which the infinitely distant point is replaced by a finite point of the plane, are valid under the corresponding conditions.

Theorem 2. *Let H and T be complete self-adjoint operators of finite orders ρ and r, respectively; let A, A_i $(i = 1, \dots, n-1)$, B_i $(i = 1, \dots, m-1)$ be completely continuous operators. Let $(E - A)^{-1}$ be bounded. Then the system of e. a. elements of the operator*

$$L(\lambda) = \sum_{i=1}^{n-1} \lambda^i A_{iH}^i + \lambda^n H^n + \sum_{i=1}^{m-1} \frac{1}{\lambda^i} B_{iT}^i + \frac{1}{\lambda^m} T^m + A$$

is $(r_i + m)$-fold complete in the space $\mathfrak{H}$.

Proof. Using Lemma 2 and Remark 1, we find that the resolvent of the operator L has finite orders, not exceeding ρ and r, at the infinitely distant point and at zero respectively. Further, pri-

applying Lemma 3, we obtain that as $|\lambda| \to \infty$ the resolvent of the operator $L(\lambda)$ remains bounded on the rays from (2), while as $|\lambda| \to 0$ on rays lying inside the angles

$$\frac{\pi k}{m} + \varepsilon \leq \arg \frac{1}{\lambda} \leq \frac{\pi(k+1)}{m} - \varepsilon \tag{2'}$$

for any $\varepsilon > 0$.

Now suppose that the theorem is false. Then there exist $n + m$ elements $f_1, f_2, \dots, f_n, \varphi_1, \varphi_2, \dots, \varphi_m$ such that

$$y(\lambda) = (E - L^*(\lambda))^{-1} \left(\sum_{i=0}^{n-1} \lambda^i f_{i+1} + \sum_{i=1}^{m} \lambda^{-i} \varphi_i \right) \tag{3}$$

has no singularities in the finite part of the plane, except perhaps at zero. As $|\lambda| \to \infty$ on the rays from (2), $y(\lambda)$ grows no faster than $|\lambda|^{n-1}$, and as $|\lambda| \to 0$ on the rays from (2′), $y(\lambda)$ grows no faster than $|\lambda|^{1-m}$.

Using Theorem 1, it is not difficult to prove that the point zero can be a pole of the function $y(\lambda)$ of order no greater than $m - 1$, and the point at infinity a pole of order no greater than $n - 1$. Consequently, the Laurent series for $y(\lambda)$ has the form

$$y(\lambda) = \sum_{i=-m}^{n-1} \lambda^i y_i. \tag{4}$$

From (3) we have

$$y(\lambda) = L(\lambda)y(\lambda) + \sum_{i=1}^{m} \lambda^{-i}\varphi_i + \sum_{i=0}^{n-1} \lambda^i f_{i+1}. \tag{5}$$

Substituting into (5) the expression for $y(\lambda)$ from (4) and comparing the coefficients of equal powers of λ, it is not difficult to show that

$$\sum_{i=1}^{m} \lambda^{-i}\varphi_i + \sum_{i=0}^{n-1} \lambda^i f_{i+1} = 0,$$

therefore $\varphi_i = 0$, $f_j = 0$ $(i = 1, \dots, n;\ j = 1, \dots, m)$. This proves the theorem.

Remark 2. From the proof of Theorem 2 it is clear that it remains valid also in the case when the operator L_2 has the form

$$L_2(\lambda) = \sum_{i=1}^{m-1} (\lambda - \lambda_0)^{-i} A_{iT}^i + (\lambda - \lambda_0)^{-m} T^m,$$

where λ_0 is any complex number, and the conditions on the operators A_i $(i = 1, 2, \dots, m-1)$ and T are the same as in Theorem 2.

Theorem 3. *Let the operator*

$$L_2(\lambda) = \sum_{i=1}^{m} \sum_{k_i=1}^{n_i} \frac{B_{i,k_i}}{(\lambda - \lambda_i)^{k_i}},$$

where B_{i,k_i} $(i = 1, \dots, m;\ k_i = 1, \dots, n_i)$ are finite-dimensional operators, and the operators H, A, A_i $(i = 1, \dots, n{-}1)$ satisfy the conditions of Theorem 2. Then the system of eigen- and associated elements of the operator $L = L_1(\lambda) + L_2(\lambda) + A$ is n-fold complete in the space $\mathcal{H}$.

The proof of this theorem is essentially analogous to the proof of Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.