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Abstract

Full Text

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THE SYMMETRIZATION PRINCIPLE FOR MULTIPLY CONNECTED DOMAINS

(Presented by Academician M. A. Lavrent'ev, 14 II 1964)

W. K. Hayman in the work ⁽¹⁾ established the so-called symmetrization principle for functions regular in a disk, which enabled Hayman, as well as other authors, to establish a number of deep properties of regular functions. In the present article we prove a symmetrization principle for multiply connected domains, of which both Hayman's principle and its generalization obtained by Abe and Kobori ⁽²⁾ are special cases.

Let G be an arbitrary domain of the z -plane bounded by a finite number of Jordan curves. Denote by $r(G, z_0)$ the inner radius of the domain G with respect to the point $z_0 \in G$ ⁽³⁾, p.97). Let $\mathfrak{R}^{(p)}(G)$ be the class of functions $\omega = f(z)$, regular in the domain G , satisfying the conditions

$$f(z_0) = \omega_0; \quad f'(z_0) = f''(z_0) = \dots = f^{(p-1)}(z_0) = 0; \quad f^{(p)}(z_0) = p!a_p \neq 0,$$

and let G_f be the image of the domain G under the mapping effected by the function $\omega = f(z)$.

Theorem 1. *If the function $\omega = f(z) \in \mathfrak{R}^{(p)}(G)$; z_k and p_k ($k = 1, 2, \dots$) are respectively the roots of the equation $f(z) - \omega_0 = 0$, distinct from z_0 , and their multiplicities, and $g_G(z, z_k)$ is the Green function of the domain G with pole at the point z_k , then*

$$r(G_f, \omega_0) \geq |a_p| r^p(G, z_0) \exp \sum_k p_k g_G(z_0, z_k). \quad (1)$$

If the domain G_f is bounded by a finite number of Jordan curves, and the total number of roots of the equation $f(z) - \omega_0 = 0$, counted with their multiplicities, is m , then equality in (1) can occur only in the case when each point of the domain G_f is the image of m points of G .

Proof. 1) Suppose first that the domain G_f is bounded by a finite number of Jordan curves and, consequently, that the Green function exists for it.

According to Lindelöf's principle (see ⁽⁴⁾, p. 91), we have:

$$g_{G_f}(\omega, \omega_0) \geq pg_G(z, z_0) + \sum_k p_k g(z, z_k),$$

i.e.

$$\begin{aligned} \log \left| \frac{(z - z_0)^p r(G_f, \omega_0)}{a_p (z - z_0)^p + a_{p+1} (z - z_0)^{p+1} + \dots} \right| + O(1) &\geq \\ &\geq p [g_G(z, z_0) + \log |z - z_0|] + \sum_k p_k g_G(z, z_k). \end{aligned}$$

Passing to the limit as z tends to z_0 , we obtain

$$r(G_f, \omega_0) \geq |a_p| r^p(G, z_0) \exp \sum_k p_k g_G(z_0, z_k).$$

Taking into account the conditions under which equality may occur in Lindelöf's principle, we also arrive at the proof of the supplementary assertion concerning when equality may occur in (1).

- 2) We shall now impose no additional restrictions on the domain G_f . In the case under consideration the inner radius $r(G_f, w_0) = \sup r(G', w_0)$, where the supremum is taken over all domains $G' \subset G_f$ bounded by a finite number of Jordan curves. If $r(G_f, w_0) = +\infty$, then inequality (1) is obvious. Let $r(G_f, w_0) < +\infty$.

Denote by $\{G_n\}$ a sequence of domains $G_1, G_2, \dots, G_n, \dots, \bar{G}_n \subset G$, each of which is bounded by a finite number of analytic Jordan curves, and such that their boundary converges uniformly to the boundary of the domain G (the latter means that for every $\varepsilon > 0$ one can indicate an N such that, if $n > N$, then each boundary point of the domain G_n is at a distance not greater than ε from the boundary of G , and conversely). We shall also assume that on the boundary of the domains G_n , $f'(z) \neq 0$. The set of values of the function $w = f(z)$ in the domain G_n will be denoted by G_n^f . By extending the domains G_n^f one can always construct a domain \tilde{G}_n^f , containing G_n^f , contained in G_f , and bounded by a finite number of Jordan curves. Applying the Lindelöf principle to the domains G_n and \tilde{G}_n^f , we again obtain:

$$r(\tilde{G}_n^f, w_0) \geq |a_p| r^p(G_n, z_0) \exp \sum_{k=1}^m p_k g_{G_n}(z_0, z_k),$$

where the summation in the exponent extends over those k for which $z_k \in G_n$.

Consequently,

$$r(G_f, w_0) \geq |a_p| r^p(G_n, z_0) \exp \sum_{k=1}^m p_k g_{G_n}(z_0, z_k). \quad (2)$$

Passing in inequality (2) to the limit as $n \rightarrow \infty$, we obtain

$$r(G_f, w_0) \geq |a_p| r^p(G, z_0) \exp \sum_{k=1}^m p_k g_{G_n}(z_0, z_k).$$

Taking into account that the last inequality is valid for arbitrary m (if the number of roots is infinite), we arrive at (1).

Remark. It can be shown that inequality (1) (and also (4)) remains valid also in the case of an arbitrary domain G . The Green function in this case is defined by means of an approximating sequence of domains (see, for example, (4), p. 103). If for the domain G the Green function does not exist, then it does not exist for the domain G_f either.

Let us establish the connection between the inner radius of the domain G at the point $z = z_0$ and the Ahlfors function ⁽⁵⁾, which maps the domain G onto a full n -sheeted disk and solves the extremal problem of finding $\sup f'(z_0)$ in the class of functions regular in the domain G and satisfying the conditions

$$f(z_0) = 0; \quad |f(z)| < 1, \quad z \in G; \quad f'(z_0) > 0.$$

Denote the above-mentioned Ahlfors function by $w = F_G(z, z_0)$. It is known (see, for example, ⁽⁵⁾, p. 524) that

$$|F_G(z, z_0)| = \exp \sum_{k=0}^{n-1} -g_G(z, z_k),$$

where $z_1, z_2, \dots, z_{n-1}, z_0$ are the zeros of the function $F_G(z, z_0)$, each of them being taken as many times as its multiplicity.

Hence

$$\log \left| \frac{z - z_0}{F_G(z, z_0)} \right| = g_G(z, z_0) + \log |z - z_0| + \sum_{k=1}^{n-1} g_G(z, z_k).$$

Passing to the limit as $z \rightarrow z_0$, we obtain

$$r(z_0, G) = \frac{1}{F'_G(z_0, z_0)} \exp \sum_{k=1}^m -g_G(z_0, z_k). \quad (3)$$

If G is a simply connected domain, then equality (3) corresponds to the well-known definition of the inner (conformal) radius, since in this case the roots z_k , $k = 1, 2, \dots$, are absent, and the function $w = F_G(z, z_0)$ maps the domain G univalently onto the unit disk with center at the origin.

Combining Theorem 1 with the symmetrization result of Pólya and Szegő, we obtain the following symmetrization principle for multiply connected domains.

Theorem 2. If the function $w = f(z) \in \mathfrak{R}^{(p)}(G)$, and G_f^* is the result of symmetrizing the domain G_f with respect to a line or a ray passing through $w = w_0$, then

$$r(G_f^*, w_0) \geq |a_p| r^p(G, z_0) \exp \sum_k p'_k g_G(z_0, z'_k), \quad (4)$$

where z'_k are the roots of the equation $f(z) = w_0$, distinct from z_0 , and p'_k are their multiplicities.

Theorem 2 makes it possible to generalize a number of known properties of functions regular in the disk to the case of functions regular in multiply connected domains, and also to improve already existing results concerning functions regular in multiply connected domains (various covering theorems, theorems on the product of conformal radii of mutually nonoverlapping domains, etc.). Moreover, the symmetrization principle for multiply connected domains is easily extended to the case of nonunivalent Q -quasiconformal mappings of multiply connected domains.

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Note: Figure translations are in progress. See original paper for figures.

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