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Abstract

Full Text

MATHEMATICS

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ON THE RIEMANN BOUNDARY-VALUE PROBLEM WITH INFINITE INDEX

(Presented by Academician P. Ya. Kochina on 10 X 1963)

As is known, the principal characteristic determining the number of linearly independent solutions of the Riemann boundary-value problem is the index of its coefficient. In the present article a solution of this problem is given for one of the cases in which the index tends to infinity.

§ 1. **Definition 1.** If on the half-interval $a \leq t < b$, $-\infty < a < b \leq \infty$, a continuous function $G(t) \neq 0$ is given and $\lim_{t \rightarrow b-0} \arg G(t) = +\infty$ (or $-\infty$), then we shall say that $\text{Ind } G(t) = +\infty$ (or $-\infty$), and that $t = b$ is a point of positive (negative) left-hand winding of the function $G(t)$. A point of right-hand winding is defined analogously. In the present article only the case of positive winding is considered.

Definition 2. Let inside the angle $\alpha < \arg z < \beta$ there be given an analytic function, continuous in $\alpha \leq \arg z \leq \beta$ (for $z \neq \infty$), and suppose that for some $\mu > 0$ the estimate

$$|f(z)| < \exp(|z|^\mu) \quad \text{for } |z| > R. \quad (1)$$

holds. Then the exact lower bound ρ of all numbers $\nu > 0$ for which

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln |f(re^{i\theta})|}{r^\nu} \equiv 0 \quad \text{for } \alpha \leq \theta \leq \beta, \quad (2)$$

is called the order of the function $f(z)$, and the function

$$h_f(\theta) = \overline{\lim}_{r \rightarrow \infty} \frac{\ln |f(re^{i\theta})|}{r^\rho} \quad (3)$$

is called its indicator.

If $\inf_\nu \{\nu\} = \inf\{\mu\}$ (as, for example, for an entire function considered in the whole plane), then this definition of order is equivalent to the generally accepted one ((², p. 69). It is introduced mainly for characterizing the decrease of a

function bounded inside an angle. For example, if $f(z) = e^{-z}$, $-\pi/2 \leq \arg z \leq \pi/2$, then $\rho = 1$, $h_f(\theta) = -\cos \theta$. The validity of the definition just made follows from the fact that:

1. If $f(z)$ satisfies estimate (1), then (2) is certainly fulfilled for

$$\nu > \max \left\{ \mu, \frac{\pi}{\beta - \alpha} \right\}.$$

2. For any $\rho_0 \geq 0$, $\alpha < \beta$, one can construct an example of a function $f(z)$, analytic in the angle $\alpha \leq \arg z \leq \beta$, for which $\inf\{\nu\} = \rho_0$, but, nevertheless, (1) is not fulfilled for any $\mu > 0$.
3. The indicator (3) has the same properties as in the generally accepted case ((², pp. 72-85), since from (1) and (2) it can be inferred that $|f(z)| < \exp(|z|^{\rho+\varepsilon})$ for $|z| > R(\varepsilon)$, $\varepsilon > 0$.

Definition 3. If on the half-interval $[a, \infty)$, $a > 0$, a function $f(t)$ is given, then we shall say that it satisfies the Hölder condition: $f(t) \in H(\mu)$, if

$$|f(t_1) - f(t_2)| < A \left| \frac{1}{t_1} - \frac{1}{t_2} \right|^\mu, \quad 0 < \mu \leq 1, \quad A, \mu = \text{const.}$$

§ 2. Denote by D the domain with boundary $L = [1, \infty]$. Consider in the domain D the homogeneous Riemann boundary-value problem

$$\Phi^+(t) = G(t)\Phi^-(t) \tag{4}$$

under the following assumptions:

1. $\arg G(t) = \varphi(t)t^\rho$, $0 < \rho < \frac{1}{2}$, $\varphi(t) \in H(\mu)$, $0 < \mu \leq 1$, $\varphi(\infty) = \lambda > 0$, $-2\pi < \arg G(1) \leq 0$.
2. $\ln|G(t)| \in H(\mu)$.

Obviously, $\text{Ind } G(t) = +\infty$, and $t = \infty$ is a point of twisting. The number ρ will be called the **order of twisting**, and λ its coefficient. The point of twisting has been transferred to infinity for the convenience of introducing into the subsequent investigation the apparatus of entire functions.

We shall solve problem (4) in the class of bounded functions analytic in D , continuous up to $\tilde{L} = (1, \infty)$, and satisfying on \tilde{L} the boundary condition (4). We shall call a **canonical function of the boundary-value problem** (4) such a bounded solution $X(z)$ of it which has no zeros in $D + \tilde{L}$, is normalized by the condition $X(0) = 1$, and in a neighborhood of the point $z = 1$ satisfies the estimate

$$|X(z)| > C|z - 1|^\alpha, \quad \text{where } 0 \leq \alpha < 1, \quad C > 0; \quad C, \alpha = \text{const.}$$

It can be proved that the canonical function is expressed by the formula

$$X(z) = \exp \left[\frac{z}{2\pi i} \int_1^\infty \frac{\ln G(x)}{x(x-z)} dx \right]. \quad (5)$$

Theorem 1. *The canonical function has in the domain D order ρ with indicator*

$$h_X(\theta) = \lim_{r \rightarrow \infty} \frac{\ln |X(re^{i\theta})|}{r^\rho} = -\frac{\lambda}{2 \sin \rho\pi} \cos \rho(\theta - \pi), \quad 0 \leq \theta \leq 2\pi,$$

and the convergence to the limit is uniform in θ on the interval $[0, 2\pi]$.

Theorem 2. *The homogeneous boundary-value problem (4) with infinite index under assumptions 1 and 2 has an infinite set of linearly independent bounded solutions, whose general formula has the form*

$$\Phi(z) = F(z)X(z) = cz^m \prod_{n=1}^{n_0} \left(1 - \frac{z}{z_n} \right) \exp \left[\frac{z}{2\pi i} \int_1^\infty \frac{\ln G(x) dx}{x(x-z)} \right], \quad n_0 \leq \infty, \quad (6)$$

where $F(z)$ is an entire function whose order ρ_F (or the exponent of convergence γ_F of the sequence of its zeros $\{z_n\}$) does not exceed ρ , and, moreover, for sufficiently large $t > 0$ the inequality holds

$$\ln |F(t)| < \frac{\lambda}{2} \operatorname{ctg} \rho\pi \cdot t^\rho - \frac{t}{2\pi} \int_1^\infty \frac{\varphi(x) - \lambda}{x^{1-\rho}(x-t)} dx + C_F, \quad C_F = \text{const.} \quad (7)$$

Remark. The integral in (7), for $t > t_0(\varepsilon)$, admits the estimate

$$\left| \frac{t}{2\pi} \int_1^\infty \frac{\varphi(x) - \lambda}{x^{1-\rho}(x-t)} dx \right| < t^{\rho-\mu+\varepsilon} \quad \text{for any } \varepsilon > 0.$$

If the class of solutions is narrowed, then a simpler result is obtained:

Theorem 3. *The general form of the solutions of problem (4) in the class of functions of order ρ with everywhere negative indicator $h_\Phi(\theta) < h_\Phi < 0$ ($0 \leq \theta \leq 2\pi$) is given by equality (6), where: 1) either $\rho_F < \rho$ ($\gamma_F < \rho$); 2) or $\rho_F = \rho$ ($\gamma_F = \rho$),*

and in this case $h_F(0) < \frac{\lambda}{2} \operatorname{ctg} \rho\pi$. The indicator of any solution is expressed by the formula

$$h_\infty(\theta) = h_F^{(\rho)}(\theta) - \frac{\lambda}{2 \sin \rho\pi} \cos \rho(\theta - \pi), \quad 0 \leq \theta \leq 2\pi,$$

where $h_F^{(\rho)}(\theta) \equiv 0$ for $\rho_F < \rho$; $h_F^{(\rho)}(\theta) \equiv h_F(\theta)$ for $\rho_F = \rho$.

§ 3. Let us now consider, in the domain D , the nonhomogeneous Riemann boundary-value problem with infinite index

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad 1 < t < \infty, \quad (8)$$

under the following assumptions:

1. $\arg G(t) = \varphi(t)t^\rho$, $0 < \rho < 1/2$, $\varphi(t) \in H(\mu_0)$, $\rho < \mu_0 \leq 1$,
 $\varphi(\infty) = \lambda > 0$.
2. $\ln |G(t)|$, $g(t) \in H(\mu)$, $0 < \mu \leq 1$.
3. Either $\arg G(1) = 0$ and $g(1) = 0$, or $-2\pi < \arg G(1) < 0$.

We shall solve this problem in the class of bounded functions. Clearly, it suffices to find one particular solution $\Phi_0(z)$.

Let $\Psi_0(z)$ be some solution of the corresponding homogeneous problem having no zeros on \tilde{L} . Then, applying the method of F. D. Gakhov ([1], p. 117), we arrive at the formula

$$\Phi_0(z) = \frac{\Psi_0(z)}{2\pi i} \int_1^\infty \frac{g(x) dx}{\Psi_0^+(x)(x-z)}.$$

It remains to determine how to choose $\Psi_0(z)$ so that the resulting integral converges and $\Phi_0(z)$ is bounded. According to (6), $\Psi_0(z) \equiv X(z)F_0(z)$, and the problem reduces to finding the appropriate entire function $F_0(z)$. Investigating the properties of the latter, it is easy to establish that

$$\rho_{F_0} = \rho, \quad h_{F_0}(\theta) \leq \frac{\lambda}{2 \sin \rho\pi} \cos \rho(\theta - \pi), \quad h_{F_0}(0) = \frac{\lambda}{2} \operatorname{ctg} \rho\pi.$$

This function is not unique. Using certain auxiliary propositions, we find it in the class of functions with zeros on the ray $\arg z = \pi$ in the form

$$F_0(z) = z \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right), \quad r_n = \left[\frac{(2n-1)\pi}{\lambda \cos \rho\pi}\right]^{1/\rho}. \quad (9)$$

Hence, taking Theorem 2 into account, we obtain

Theorem 4. *The nonhomogeneous Riemann boundary-value problem (8) with infinite index under assumptions 1, 2, 3 has an innumerable set of bounded linearly independent solutions, whose general formula has the form*

$$\Phi(z) = X(z) \left[\frac{F_0(z)}{2\pi i} \int_1^\infty \frac{g(x) dx}{X^+(x)F_0(x)(x-z)} + F(z) \right],$$

where $F_0(x)$ is defined by (9), and $F(z)$ is an arbitrary entire function of order $\rho_F \leq \rho$, for which, for sufficiently large $t > 0$, the estimate holds:

$$\ln |F(t)| < \frac{\lambda}{2} \operatorname{ctg} \rho\pi \cdot t^\rho + C_F, \quad C_F = \text{const.}$$

It is easy to show that problem (8), generally speaking, has no solutions of order ρ with indicator $h_\Phi(\theta) \leq h_\Phi < 0$ ($0 \leq \theta \leq 2\pi$).

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REFERENCES

1. F. D. Gakhov, *Boundary-value problems*, 1963.
2. B. Ya. Levin, *Distribution of roots of entire functions*, 1956.

Note: Figure translations are in progress. See original paper for figures.

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