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Abstract

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ON THE BOUNDEDNESS OF SOLUTIONS OF SOME LINEAR PARTIAL DIFFERENCE EQUATIONS

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We consider functions $x(t_1, t_2, \ldots, t_n)$, $y(t_1, t_2, \ldots, t_n)$, … with values belonging to a complex Banach space E, defined in the domain $0 \leqslant t_1, t_2, \ldots, t_n < \infty$ and bounded in every domain $0 \leqslant t_j \leqslant b_j < \infty$, $j = 1, 2, \ldots, n$.

Introduce the "difference" operators

$$\Delta_j x = \frac{1}{\delta_j}\left[\, x(t_1, \ldots, t_{j-1}, t_j + \delta_j, t_{j+1}, \ldots, t_n) - x(t_1, \ldots, t_n)\,\right]$$

$$(\delta_j > 0,\ j = 1, 2, \ldots, n)$$

and consider the equation

$$\Delta_1^{p_1} \Delta_2^{p_2} \cdots \Delta_n^{p_n} y - \sum A_{q_1 q_2 \cdots q_n} \Delta_1^{q_1} \Delta_2^{q_2} \cdots \Delta_n^{q_n} y = x \qquad (1)$$

with "highest" term $p_j \geqslant q_j$ and $\sum p_j > \sum q_j$ for every term of $\sum$.

Let the coefficients of the equation be families of linear operators $A_{q_1 \cdots q_n} = A_{q_1 \cdots q_n}(t_1, \ldots, t_n)$, compact (in the operator norm in E) and of weak variation at infinity. The latter means that for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that, whenever $\sum t'_j \geqslant T$, $\sum t''_j \geqslant T$, $\sum |t'_j - t''_j| < 1$, always

$$\|A_{q_1 \cdots q_n}(t'_1, \ldots, t'_n) - A_{q_1 \cdots q_n}(t''_1, \ldots, t''_n)\| < \varepsilon$$

(see ([1], [2])).

For equation (1), under the natural boundary conditions:

$$y\big|_{0 \leqslant t_j < p_j \delta_j} = f_j(t_1, \ldots, t_n) \qquad (j = 1, 2, \ldots, n) \qquad (2)$$

(f_j are given functions suitably compatible with one another), the existence and uniqueness theorem is obvious. We shall give here necessary and sufficient conditions under which, to bounded (in the whole domain $0 \leqslant t_1, t_2, \dots, t_n < \infty$) functions $x(t_1, \dots, t_n)$ and $f_j(t_1, \dots, t_n)$, there always corresponds a bounded solution $y(t_1, \dots, t_n)$ of the boundary-value problem (1)–(2).

1°. The simplest problem for functions of one variable:

$$\Delta y - \lambda y = x \qquad (\lambda \text{ constant}),$$

$$y(t) = \theta \qquad \text{for } 0 \leqslant t < \delta$$

is equivalent to the equation

$$y - \lambda S y = S x,$$

where S is the operator realized on vectors–sequences $(y(t), y(t+\delta), \dots, y(t+k\delta), \dots)$ by means of the matrix

$$\delta \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 1 & 0 & 0 & \dots & 0 & \dots \\ 1 & 1 & 0 & \dots & 0 & \dots \\ 1 & 1 & 1 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \qquad (3)$$

The solution of this equation

$$y = (I - \lambda S)^{-1} S x$$

is given by the matrix

$$\delta \begin{pmatrix} 0 & 0 & 0 \dots 0 \dots & \\ 1 & 0 & 0 \dots 0 \dots & \\ 1 + \lambda\delta & 1 & 0 \dots 0 \dots & \\ (1+\lambda\delta)^2 & 1 + \lambda\delta & 1 \dots 0 \dots & \\ . & . & . & . \ . \ . \ . \ . \ . \end{pmatrix}.$$

It follows at once from this that to a bounded $x(t)$ there always corresponds a bounded $y(t)$ if and only if $|\lambda + 1/\delta| < 1/\delta$.

2°. Consider problem (1)–(2) with constant coefficients $A_{q_1 \dots q_n}$ and zero boundary conditions (for simplicity). It is equivalent to the equation

$$y - \sum S_1^{p_1 - q_1} S_2^{p_2 - q_2} \dots S_n^{p_n - q_n} A_{q_1 q_2 \dots q_n} y = S_1^{p_1} S_2^{p_2} \dots S_n^{p_n} x,$$

where the operators S_j are constructed on systems of values $\{y(t_1 + k_1\delta_1, t_2 + k_2\delta_2, \dots, t_n + k_n\delta_n)\}$ by means of matrices of the form (3). As noted in (2), the solution of such an equation may be written in the form

$$y = \left(\frac{1}{2\pi i}\right)^n \int_{\gamma_1}\int_{\gamma_2}\cdots\int_{\gamma_n} (I-\lambda_1 S_1)^{-1}\cdots(I-\lambda_n S n)^{-1}\left(I - \sum \frac{A_{q_1\cdots q_n}}{\lambda_1^{p_1-q_1}\dots\lambda_n^{p_n-q_n}}\right)^{-1} S_1^{p_1}\cdots S_n^{p_n} x\,\frac{d\lambda_1}{\lambda_1}\cdots\frac{d\lambda_n}{\lambda_n},$$

where the contours $\gamma_1, \gamma_2, \dots, \gamma_n$ are chosen so that outside them there exists a bounded operator

$$\Gamma(\lambda_1, \dots, \lambda_n) = \left(\lambda_1^{p_1}\cdots\lambda_n^{p_n} I - \sum A_{q_1\cdots q_n}\lambda_1^{q_1}\cdots\lambda_n^{q_n}\right)^{-1}. \tag{4}$$

Relying on the considerations of item 1° and using the methods developed in (2), one can establish the following.

For in the boundary-value problem (1)–(2) with constant coefficients $A_{q_1\cdots q_n}$, every bounded initial function $f_j(t_1, \dots, t_n)$ and every bounded right-hand side $x(t_1, \dots, t_n)$ to correspond always to a bounded solution $y(t_1, \dots, t_n)$ (in what follows we shall call such a boundary-value problem stable), it is necessary and sufficient that every point $(\lambda_1^0, \dots, \lambda_n^0)$, whose coordinates satisfy the inequalities

$$\left|\lambda_j^0 + \frac{1}{\delta_j}\right| \geq \frac{1}{\delta_j}, \qquad (j = 1, 2, \dots, n), \tag{5}$$

be regular for the operator-function (4), i.e., that there exist a bounded operator $\Gamma(\lambda_1^0, \dots, \lambda_n^0)$.

3°. In the case when $A_{q_1\cdots q_n} = A_{q_1\cdots q_n}(t_1, \dots, t_n)$ and the conditions formulated in the introductory paragraph are satisfied, instead of (4) one must consider the operator-functions

$$\Gamma^{(\omega)}(\lambda_1, \dots, \lambda_n) = \left(\lambda_1^{p_1}\cdots\lambda_n^{p_n} I - \sum A^{(\omega)}_{q_1\cdots q_n}\lambda_1^{q_1}\cdots\lambda_n^{q_n}\right)^{-1}. \tag{6}$$

Here $A^{(\omega)}_{q_1\cdots q_n}$ are ω-limit operators generated by the families $A_{q_1\cdots q_n}(t_1, \dots, t_n)$ simultaneously, i.e. on some common sequence of points $(t_1, \dots, t_n)$ tending to infinity.

Problem (1)–(2) is stable if and only if every point $(\lambda_1^0, \dots, \lambda_n^0)$ satisfying (5) is regular for every ω-limit operator-function (6).

4°. The domain defined by inequality (5) is the exterior of the circle with center $-1/\delta_j$ and radius $1/\delta_j$; as $\delta_j \to 0$ it becomes the right half-plane. Therefore the main result ([2])—the stability criterion for the boundary-value problem

$$\frac{\partial^{p_1+\cdots+p_n} y}{\partial t_1^{p_1}\cdots\partial t_n^{p_n}} - \sum A_{q_1\cdots q_n} \frac{\partial^{q_1+\cdots+q_n} y}{\partial t_1^{q_1}\cdots\partial t_n^{q_n}} = x,$$

$$\left.\frac{\partial^k y}{\partial t_j^k}\right|_{t_j=0} = f_j; \qquad k = 0, 1, \dots, p_j - 1; \qquad j = 1, 2, \dots, n, \tag{7}$$

is included in item 3° as a limiting case.

Comparing the results obtained for the difference and differential boundary-value problems (with the same coefficients $A_{q_1\cdots q_n}$), one easily sees that:

If for some $\delta_j > 0$ problem (1)–(2) is stable, then it remains stable for all $0 \leq \delta_j' \leq \delta_j$ ($j = 1, 2, \dots, n$); in particular, problem (7) is stable.

If problem (7) is stable, then there exist $\delta_j^0 > 0$ such that for all $\delta_j < \delta_j^0$ ($j = 1, 2, \dots, n$) problem (1)–(2) is also stable.

The last considerations may be useful in investigations connected with the application of the method of grids to the solution of boundary-value problems in an unbounded domain.

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REFERENCES

1. M. G. Krein, UMN, 3, issue 3 (25), 166 (1948).

2. M. A. Rutman, DAN, 147, No. 4, 789 (1962).

Note: Figure translations are in progress. See original paper for figures.