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Abstract

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On the Multiplicity of the Spectrum of a Self-Adjoint Ordinary Differential Operator

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In the present paper we study the multiplicity of the spectrum of a self-adjoint operator A, generated in the Hilbert space $\mathcal{L}^2(a,b)$ by a formally self-adjoint ordinary differential operation l of arbitrary even order $2n$. Here one or both endpoints of the interval (a,b) may be singular. As is known, in the case of an interval with two singular endpoints the multiplicity of the spectrum of the operator A does not exceed $2n$; if at least one endpoint is regular and the operator A is defined by separated boundary conditions, then the multiplicity of its spectrum does not exceed n. Quite often, however, the multiplicity of the spectrum of a differential operator of order $2n$ turns out to be smaller than the bounds indicated above. The present paper is devoted to the study of some such cases. We estimate the multiplicity of the part of the spectrum of the operator A contained in the segment $[\alpha,\beta]$, proceeding from the assumption that the differential equation $l[y] = \lambda y$ has, for every $\lambda \in [\alpha,\beta]$, solutions possessing certain properties in a neighborhood of a singular endpoint of the interval (a,b). Since a number of assertions are known which make it possible to judge the asymptotics of solutions of a differential equation from the behavior of its coefficients, the results of the article may in some cases prove useful for determining the multiplicity of the spectrum of a differential operator on the basis of the properties of the coefficients of the expression $l[y]$.*

1. Let us recall that a formally self-adjoint differential expression $l[y]$ of order $2n$ can be represented in the form

$$l[y] = p_n y - \frac{d}{dx}\left\{p_{n-1}\frac{dy}{dx} - \frac{d}{dx}\left[p_{n-2}\frac{d^2y}{dx^2} - \cdots - \frac{d}{dx}\left(p_0\frac{d^ny}{dx^n}\right)\cdots\right]\right\};$$

it is assumed that the functions

$$p_0^{-1}(x),\ p_1(x),\ \ldots,\ p_n(x) \tag{1}$$

are real and summable on every segment $[a_1,b_1] \subset (a,b)$. The interval (a,b) may also be infinite. If the point a or b is finite and the functions (1) are summable in its one-sided neighborhood, i.e., if this endpoint is regular, then we agree to include it in the interval (a, b).

To each function $y(x)$ for which $l[y]$ has meaning, we associate the vector-function

$$\hat{y}(x) = (y^{[0]}(x),\, y^{[1]}(x),\, \dots,\, y^{[2n-1]}(x)),$$

where $y^{[0]}(x) = y(x)$, and $y^{[k]}(x)$ $(k = 1, 2, \dots, 2n-1)$ denotes the so-called

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* The study of the multiplicity of the spectrum of a self-adjoint differential operator of second order is the subject of the works of I. S. Kac ([1,2]). They completely clarify the question of how the multiplicity of the spectrum of a differential operator in the space $\mathcal{L}^2(a, b)$ depends on the properties of the spectral functions of differential operators generated by the same differential expression in the spaces $\mathcal{L}^2(a, c)$ and $\mathcal{L}^2(c, b)$, where c is an arbitrary interior point of the interval (a, b). M. G. Krein and I. S. Kac observed that the author' s results ([3]), concerning the spectral theory of symmetric non-self-adjoint differential operators of second order, allow one to draw certain conclusions about the multiplicity of the spectrum of a self-adjoint differential operator. This is what attracted the author' s attention to the range of questions considered here.

the quasi-derivative of order k of the function $y(x)$, we shall regard $\hat{y}(x)$, for every $x \in (a, b)$, as a column matrix. Introduce the square matrix $J = \|\varepsilon_{jk}\|_1^{2n}$, putting $\varepsilon_{jk} = 0$ if $j + k \neq 2n + 1$, $\varepsilon_{jk} = \operatorname{sign}(j - k)$ if $j + k = 2n + 1$ $(j, k = 1, 2, \dots, 2n)$. For any functions $u(x)$ and $v(x)$ to which the operation l is applicable, the well-known Lagrange identity holds

$$l[u]\bar{v} - u\overline{l[v]} = \frac{d}{dx}\,(\hat{v}^* J \hat{u})\,,$$

where the asterisk denotes passage to the conjugate matrix, in the present case a one-row matrix.

Consider the closed symmetric operator $\mathcal{L}_0$ with minimal domain of definition, generated in $\mathcal{L}^2(a, b)$ by the operation l^*. As is known, this operator has defect index (r, r), where $0 \leq r \leq 2n$. Let A be a self-adjoint operator in $\mathcal{L}^2(a, b)$ with domain of definition D_A, which is an extension of the operator $\mathcal{L}_0$, and let E_λ $(-\infty < \lambda < \infty)$ be the spectral function of the operator A.

By

$$y_1(x;\lambda),\; y_2(x;\lambda), \dots,\; y_{2n}(x;\lambda)$$

we denote a fundamental system of solutions of the equation

$$l[y] = \lambda y, \tag{2}$$

which satisfy the initial conditions:

$$y_k^{[j-1]}(x_0; \lambda) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \qquad (j, k = 1, 2, \ldots, 2n)$$

where x_0 is an arbitrary fixed point of the interval (a, b). As is known**, for any real values α and β the operator

$$\frac{E_{\beta-0} + E_{\beta+0}}{2} - \frac{E_{\alpha-0} + E_{\alpha+0}}{2}$$

is integral, and its kernel $K_{\alpha,\beta}(x, s)$ is representable in the form

$$K_{\alpha,\beta}(x, s) = \int_\alpha^\beta \sum_{j,k=1}^{2n} y_k(x; \lambda) y_j(s; \lambda) \, d\rho_{jk}(\lambda),$$

where $\rho_{jk}(\lambda)$ $(j, k = 1, 2, \ldots, 2n)$ are the elements of the Hermitian nondecreasing matrix-function $T(\lambda) = \|\rho_{jk}(\lambda)\|_1^{2n}$, called the spectral distribution function of the operator A.

Lemma. Suppose that for every $\lambda \in [\alpha, \beta]$ equation (2) has a solution $v(x; \lambda)$ such that:

1)
$$\int_a^b |v(x; \lambda)|^2 \, dx < \infty,$$
where c is some interior point of the interval (a, b);

2) for every function $f(x) \in D_A$
$$\hat{f}^*(x) J \hat{v}(x; \lambda)\Big|_{x=b} = 0;$$

3) for fixed $x \in (a, b)$ the vector-function $\hat{v}(x; \lambda)$ satisfies a Lipschitz condition with respect to λ on the segment $[\alpha, \beta]$***.

* See, for example, (4) or (5).

** See, for example, (5), p. 204.

*** If condition 3) is satisfied for some $x_1 \in (a, b)$, then it is also satisfied for any other fixed $x \in (a, b)$.

Then for any $\mu_1, \mu_2 \in [\alpha, \beta]$

$$\int_{\mu_1}^{\mu_2} \hat{v}^*(x_0;\lambda) J \, dT(\lambda) = 0.$$

Let us note that the proof of this lemma is based on the results of paper ([6]) (see also ([7])).

Theorem 1. Suppose that for every $\lambda \in [\alpha, \beta]$ equation (2) has $m = m' + m''$ linearly independent solutions

$$v_1(x;\lambda),\ v_2(x;\lambda), \dots,\ v_m(x;\lambda) \tag{3}$$

such that:

1) for each of the first m' solutions (3):

a)
$$\int_a^c |v_k(x;\lambda)|^2 \, dx < \infty \quad (a < c < b);$$

b)
$$\hat{f}^*(x) J \hat{v}_k(x;\lambda)\big|_{x=a} = 0,$$

whatever the function $f(x) \in D_A$ may be;

2) for each of the last m'' solutions (3):

a)
$$\int_c^b |v_k(x;\lambda)|^2 \, dx < \infty \quad (a < c < b);$$

b)
$$\hat{f}^*(x) J \hat{v}_k(x;\lambda)\big|_{x=b} = 0,$$

whatever the function $f(x) \in D_A$ may be;

3) each of the vector-functions $v_k(x;\lambda)$, for fixed $x \in (a, b)$, satisfies a Lipschitz condition with respect to λ on the segment $[\alpha, \beta]$.

Then the multiplicity of the part of the spectrum of the operator A contained in the segment $[\alpha, \beta]$ does not exceed $2n - m$.

Remark 1. If any one of the operators with minimal domain of definition generated by the operation l in the spaces $\mathcal{L}^2(a, c)$ and $\mathcal{L}^2(c, b)$ has defect index (n, n), then in the statement of the theorem one may omit, respectively, condition 1b) or 2b), since its fulfillment is ensured in this case by condition 1a) or 2a). If, in particular, the operator $\mathcal{L}_0$ with minimal domain of definition, generated by the operation l in the space $\mathcal{L}^2(a, b)$, is self-adjoint, then both conditions 1b) and 2b) turn out to be superfluous.

Remark 2. Suppose that the operator $\mathcal{L}_0$ with minimal domain of definition $D_{\mathcal{L}_0}$, generated in $\mathcal{L}^2(a,b)$ by the operation l, has defect index (r,r), where $r > 0$. Choose in D_A an arbitrary system of functions $f_1(x), \dots, f_r(x)$, linearly independent modulo $D_{\mathcal{L}_0}$. Then in the formulation of conditions 1b) and 2b) of Theorem 1 it is enough to require that they be fulfilled only for the functions $f(x) = f_j(x)$ $(j = 1, \dots, r)$, since it already follows from this that these conditions are fulfilled for any function $f(x) \in D_A$.

2. Let us pass to the consideration of the case where the endpoint a of the interval (a,b) is regular. The defect number r of the operator $\mathcal{L}_0$ with minimal domain of definition, which is generated in $\mathcal{L}^2(a,b)$ by the operation l, now satisfies the inequality $n \le r \le 2n$.

Assume that the self-adjoint operator A in $\mathcal{L}^2(a,b)$, which is an extension of the operator $\mathcal{L}_0$, is defined by separated boundary conditions. Then the system of boundary conditions at the point a can be written in the form $U\hat{y}(a) = 0$, where U is some rectangular matrix consisting of n linearly independent rows and $2n$ columns, such that $UJU^* = 0$.

Theorem 2. Suppose that for every $\lambda \in [\alpha, \beta]$ equation (2) has q linearly independent solutions

$$v_1(x;\lambda), \dots,\ v_q(x;\lambda),$$

such that:

1) $\hat{v}_k(x;\lambda) \in \mathcal{L}^2(a,b)$ $(k = 1, \dots, q)$;
2) $\hat{f}^*(x) J \hat{v}_k(x;\lambda)\big|_{x=b} = 0$ $(k = 1, \dots, q)$ for every function $f(\lambda) \in D_A$;
3) the linear combination
$$\sum_{k=1}^{q} c_k v_k(x;\lambda)$$
satisfies the system of boundary conditions at the point a only when $c_1 = \dots = c_q = 0$;
4) each of the vector-functions $\hat{v}_k(x;\lambda)$, for fixed $x \in (a,b)$, satisfies a Lipschitz condition with respect to λ on the segment $[\alpha, \beta]$.

Then the multiplicity of the part of the spectrum of the operator A contained in the segment $[\alpha, \beta]$ does not exceed $n - q$.*

Remark. Taking into account conditions 1), 2), condition 3) can also be formulated in the following way: among the linear combinations

$$\sum_{k=1}^{q} c_k v_k(x;\lambda)$$

there are no eigenfunctions of the operator A. If the segment $[\alpha, \beta]$ contains no eigenvalues of the operator A, then condition 3) may be omitted.

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CITED LITERATURE

[1] I. S. Kats, *DAN*, **145**, No. 3, 510 (1962).
[2] I. S. Kats, *Izv. AN SSSR, ser. matem.*, **27**, No. 5, 1081 (1963).
[3] A. V. Shtraus, *Izv. AN SSSR, ser. matem.*, **20**, No. 6, 783 (1956).
[4] N. I. Akhiezer, I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Moscow—Leningrad, 1950.
[5] M. A. Naimark, *Linear Differential Operators*, Moscow, 1954.
[6] A. V. Shtraus, *Volga Mathematical Collection, Theoretical Series*, Kuibyshev, 1963, p. 221.
[7] A. V. Shtraus, *DAN*, **152**, No. 3, 563 (1963).

* From conditions 1), 2), and also from condition 3), the inequality $q \leq n$ follows.

Note: Figure translations are in progress. See original paper for figures.