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Abstract

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V. V. BAKLAN

EQUATIONS IN VARIATIONAL DERIVATIVES AND MARKOV PROCESSES IN HILBERT SPACE

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In this paper stochastic equations in Hilbert space are considered. For the mathematical expectations of functionals of solutions of these equations, equations in variational derivatives are obtained, analogous to Kolmogorov' s backward equations for diffusion processes in finite-dimensional space. The results obtained are at the same time an existence theorem for the solution of the Cauchy problem for equations in variational derivatives, analogous to equations of parabolic type.

Let H be a separable Hilbert space (the elements of this space will hereafter be denoted by $\xi, \varphi, \psi, ...$); A a measurable space, on whose σ-algebra of subsets a measure $\nu(d\alpha)$ $(\alpha \in A)$ is defined; $a(\tau, \varphi)$ and $b(\tau, \varphi, \alpha)$ some functions, defined on the products of spaces $[t_0, T] \times H$ and $[t_0, T] \times H \times A$, respectively, taking values in H and such that for all $\varphi, \psi \in H$ the scalar products $(a(\tau, \varphi), \psi)$ and $(b(\tau, \varphi, \alpha), \psi)$ are measurable functions of τ and α.

Consider the differential stochastic equation in H

$$d\xi(t) = D(t)\xi(t)\,dt + a(t, \xi(t))\,dt + \int_A b(t, \xi(t), \alpha)\,W(d\alpha \times dt); \tag{1}$$

here $W(d\alpha \times dt)$ is a Wiener random measure, defined on the σ-algebra $\mathfrak{A}_T$ of measurable sets from $[t_0, T] \times A$ and possessing the properties:

a) $W(d\alpha \times dt)$ takes independent values on disjoint sets;

b) if $a = \bigcup_{i=1}^{\infty} a_i$, $a_i \in \mathfrak{A}_T$, and the a_i are pairwise disjoint, then the series $\sum_{i=1}^{\infty} W(a_i)$ converges in probability to $W(a)$;

c) $W(d\alpha \times dt)$ has normal distribution $(0, \nu(d\alpha) \times dt)$.

With respect to the linear operator $D(t)$, assume that the Cauchy problem

$$\frac{d\xi(t)}{dt} = D(t)\xi(t) + \psi(t), \qquad \xi(t_0) = \varphi$$

has a unique weak solution for any $\varphi \in H$, $\psi(t) \in H$, and that this solution is representable in the form

$$\xi(t) = \Gamma_{t_0}^t \varphi + \int_{t_0}^t \Gamma_s^t \psi(s)\, ds,$$

where the linear operators Γ_s^t possess the properties:

a) $\Gamma_{t_0}^t \varphi = \Gamma_s^t \Gamma_{t_0}^s \varphi,\ \varphi \in H,\ t_0 \le s \le t \le T$;

b) $\Gamma_t^t \varphi = \varphi,\ \varphi \in H,\ t \in [t_0, T]$.

By a solution of equation (1) we shall mean a solution of the equation

$$\xi(t) = \Gamma_{t_0}^t \varphi + \int_{t_0}^t \Gamma_s^t a(s, \xi(s))\, ds + \int_{t_0}^t \int_A \Gamma_s^t b(s, \xi(s), \alpha)\, W(d\alpha \times ds). \qquad (2)$$

The second integral on the right-hand side of equation (2) is stochastic with respect to the Wiener random measure $W(d\alpha \times ds)$. Let $\mathfrak{F}_t$ be the σ-algebra generated by events of the form $\{W(a) \in \gamma\}$, where $a \in \mathfrak{A}_t$, γ is a Borel set on the line $(-\infty, \infty)$. Consider the linear space H_k of functions $\xi(t)$, for every $t \in [t_0, T]$ measurable with respect to $\mathfrak{F}_t$, with probability 1 (with respect to the probability measure corresponding to the Wiener random measure $W(d\alpha \times dt)$) taking values in H and possessing the property

$$\max_{t_0 \le t \le T} M\|\xi(t)\|^{2k} < \infty$$

($\|\xi(t)\|$ is the norm of $\xi(t)$ in H; k is some natural number). We turn H_k into a normed space by setting the norm equal to

$$|\xi(t)| = \max_{t_0 \le t \le T} \{M\|\xi(t)\|^{2k}\}^{1/2k}.$$

In the author’ s paper (1) it was proved that if the conditions are satisfied:

1) $\max_{t_0 \le t \le T} \|\Gamma_{t_0}^t\| < \infty$;

2) there exists a constant L such that for all $\varphi, \psi \in H$

$$\|a(t, \varphi) - a(t, \psi)\| \le L\|\varphi - \psi\|,$$

$$\|b(t, \varphi, \alpha) - b(t, \psi, \alpha)\| \le L\|\varphi - \psi\|,$$

then equation (2) has a unique solution in H_k.

The solution of equation (2) at time t may be regarded as the result of the action of a certain operator $S_{t_0}^t$, defined on H and taking values in H_k, on the initial value φ:

$$\xi(t) = S_{t_0}^t[\varphi].$$

Definition 1. An operator S, acting from H into H_k, is differentiable at the point $\varphi \in H$ if there exists a linear operation $U_\varphi[\cdot]$, also acting from H into H_k, such that for all $\delta\varphi \in H$

$$\left|S[\varphi + \delta\varphi] - S[\varphi] - U_\varphi[\delta\varphi]\right| \le f(\|\delta\varphi\|)$$

and $\lim\limits_{x\to 0} x^{-1} f(x) = 0$.

Definition 2. An operator S, acting from H into H_k, is n-times differentiable if there exist such multilinear operations $U_\varphi^{(i)}[\cdot, \dots, \cdot]$, acting from $H \times \cdots \times H$ $(i = 1, \dots, n)$ into H_k, that for all $\delta\varphi \in H$

$$\left|S[\varphi + \delta\varphi] - S[\varphi] - \sum_{i=1}^{n} \frac{1}{i!} U_\varphi^{(i)}[\delta\varphi, \dots, \delta\varphi]\right| \le f(\|\delta\varphi\|),$$

and $\lim\limits_{x\to 0} x^{-n} f(x) = 0$. The operation $U_\varphi^{(i)}[\cdot, \dots, \cdot]$ will be called the i-th variational derivative of the operator S and will be denoted by

$$\frac{\delta^i S}{\delta\varphi^i}[\cdot, \dots, \cdot].$$

Theorem 1. If the operators $a(t, \cdot)$, $b(t, \cdot, \alpha)$ are three times differentiable at every point $\varphi \in H$ and

$$\left\|\frac{\delta^i a}{\delta\varphi^i}\right\| \le L < \infty \qquad (i = 1, 2, 3),$$

$$\left\|\frac{\delta^i b}{\delta\varphi^i}\right\| \le L < \infty \qquad (i = 1, 2, 3)$$

for all $t \in [t_0, T]$, $\alpha \in A$, $\varphi \in H$, then the operator $S_{t_0}^t$ is twice differentiable at every point $\varphi \in H$.

The idea of the proof of this theorem belongs to I. I. Gikhman and was used by him in the paper (^{3}) to prove differentiability of solutions of stochastic differential equations in finite-dimensional space with respect to the initial data.

It can be proved that the operator $S_{t_0}^t$ has the properties of a dynamical system in the sense that

$$P\{S_s^t[S_{t_0}^s[\varphi]] = S_{t_0}^t[\varphi]\} = 1, \qquad \varphi \in H, \quad t_0 \le s \le t \le T.$$

Since the random measure $W(d\alpha \times dt)$ has independent increments with respect to t, using Theorem 1 of (^{2}) we conclude that the random process $S_{t_0}^t[\varphi]$ has the Markov property: for any bounded measurable functional $f[\cdot]$

$$Mf[S_{t_0}^t[\varphi]] = M\{Mf[S_\tau^t[\psi]] \mid S_{t_0}^\tau[\varphi] = \psi\}, \qquad t_0 < \tau < t. \tag{3}$$

Consider the functional $F(\tau, \varphi, t) = Mf[S_\tau^t[\varphi]]$. With respect to the functional $F(\tau, \varphi, t)$ we have

Theorem 2. If the functional $f[\varphi]$ is three times differentiable and is uniformly bounded in $\varphi \in H$ together with its three variational derivatives, then the functional $F(\tau, \varphi, t)$ is twice differentiable.

The proof of this theorem is based on the following lemma.

Lemma 1. Under the conditions of Theorem 1 there exists a constant C such that

$$|S_{t_0}^t[\varphi] - S_{t_0}^t[\psi]|^4 \le C\|\varphi - \psi\|^4$$

for all $\psi, \varphi \in H$.

Theorem 3. If the conditions of Theorems 1-2 are fulfilled, then the functional $F(\tau, \varphi, t)$, for all φ belonging to the domain of definition $D(\tau)$, satisfies the equation

$$\frac{\partial}{\partial \tau} F(\tau, \varphi, t) + \frac{\delta F}{\delta \varphi}[D(\tau)\varphi + a(\tau, \varphi)] +$$

$$+\frac{1}{2}\int_A \frac{\delta^2 F}{\delta \varphi^2}[b(\tau, \varphi, \alpha),\, b(\tau, \varphi, \alpha)]_\nu(d\alpha) = 0; \tag{4}$$

$$\lim_{\tau \to t-0} F(\tau, \varphi, t) = f(\varphi). \tag{5}$$

In the proof of this theorem one uses: equality (3), Theorems 1-2, and the following lemmas.

Lemma 2. Under the conditions of Theorem 3

$$\lim_{\Delta \to 0} \frac{1}{\Delta} M[S_{\tau-\Delta}^\tau[\varphi] - \varphi] = D(\tau)\varphi + a(\tau, \varphi).$$

Lemma 3. Under the conditions of Theorem 3, for any bounded bilinear operation $U[\cdot, \cdot]$,

$$\lim_{\Delta \to 0} \frac{1}{\Delta} MU \left[S_{\tau-\Delta}^{\tau}[\varphi] - \varphi,\ S_{\tau-\Delta}^{\tau}[\varphi] - \varphi\right] = \int_A U\left[b(\tau, \varphi, \alpha),\ b(\tau, \varphi, \alpha)\right] \nu(d\alpha).$$

Lemma 4. Under the conditions of Theorem 3, there exists a constant C such that

$$M\left|S_{\tau-\Delta}^{\tau}[\varphi] - \varphi\right|^4 \leq C\Delta^2.$$

The following assertion is an immediate consequence of Theorem 3.

If the linear operator $D(t)$ has the properties listed above, and the operators $a(\tau, \varphi)$, $b(\tau, \varphi, \alpha)$ and the functional $f(\varphi)$ satisfy the conditions of Theorems 1-3, then the equation in variational derivatives (4) with initial condition (5) has a solution representable in the form $Mf\left[S_t^t[\varphi]\right]$, where $S_t^t[\varphi]$ is the solution of the stochastic equation (2).

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Kiev State University

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