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Abstract

Full Text

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More on Mahler' s Hypothesis

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In the preceding note ([1]) on the same topic, a scheme was set forth for proving K. Mahler' s hypothesis on the measure of the set of complex S-numbers:

$$\sup_{(n)} \eta_n = \frac{1}{2} \qquad (n = 1, 2, \dots).$$

In this note we show how a certain modification of the previous arguments leads to a proof of the real case of K. Mahler' s hypothesis:

$$\sup_{(n)} \theta_n = 1 \qquad (n = 1, 2, \dots). \tag{1}$$

Let $\mathfrak{P}_n(h)$ be the set of irreducible primitive polynomials with integer coefficients

$$P = a_0 + a_1 x + \dots + a_{nx}^n$$

satisfying $n \geqslant 3$,

$$\max(|a_0|, |a_1|, \dots, |a_{n-1}|) \leqslant a_n = h. \tag{2}$$

Denote by $x_1, x_2, \dots, x_n$ the roots of the polynomial P, so that

$$P(x) = h(x - x_1) \cdots (x - x_n).$$

Lemma 1. Let $P \in \mathfrak{P}_n(h)$, let ω be a real number, $|P(\omega)| < h^{-w}$, $w > 0$. Put

$$|\omega - x_1| = \min_{(i)} |\omega - x_i| \qquad (i = 1, 2, \dots, n).$$

Then

$$|\omega - x_1| < c(n) h^{-1-(2w-n)/3} |D(P)|^{-1/6}.$$

Lemma 2. Suppose that, under the conditions of the preceding lemma, $w \geqslant n-1$, and x_2 is the root of the polynomial P nearest to x_1.

Then

$$\omega - x_1 \asymp \begin{cases} |P(\omega)| : P'(x_1), & \text{if } |\omega - x_1| \leqslant 2|x_1 - x_2|, \\ \left(|P(\omega)|\,|x_1 - x_2| : |P'(x_1)|\right)^{1/2}, & \text{if } |\omega - x_1| > 2|x_1 - x_2|. \end{cases}$$

Lemma 3. Let Δ be a measurable set on the line, $\operatorname{mes}\Delta < \varepsilon$, and let a system

$$\Lambda = \bigcup_{i=1}^{\infty} \lambda_i$$

of intervals λ_i be given with the conditions

$$\operatorname{mes}(\lambda_i \cap \Delta) \geqslant \frac{1}{2} \operatorname{mes}\lambda_i \qquad (i = 1, 2, \ldots).$$

Then

$$\operatorname{mes}\Lambda < 4\varepsilon.$$

We proceed to the proof of (1). Let $w_n = n\theta_n$ $(n = 1, 2, \ldots)$. Obviously, we may assume that $|\omega| < c$. Let $\sigma(P)$ be a system of pairwise nonintersecting intervals $\sigma_i(P)$, in which the inequality

$$|P(\omega)| < h^{-w}, \qquad w = w_{n-1} + \delta, \tag{3}$$

is satisfied.

where $\delta > 0$ is an arbitrary small number, but fixed in what follows; $\sigma(P) = \sum \sigma_i(P)$. Arguing as in the case of complex numbers, on the basis of Lemma 3 we conclude that only those intervals $\sigma_i(P)$ are of interest in which the set of points ω belonging to other systems $\sigma(Q)$, $Q \in \mathfrak{P}_n(h)$, has measure not less than $\frac{1}{2}\operatorname{mes}\sigma_i(P)$.

Let $\sigma_1(P)$ be one of the intervals of the system $\sigma(P)$. Define the root χ_1 of the polynomial P for which in $\sigma_1(P)$ there is at least one point ω satisfying the condition

$$|\omega - \chi_1| = \min |\omega - \chi_i| \qquad (i = 1, 2, \ldots, n).$$

Arrange the remaining roots of the polynomial P in the order $\chi_2, \chi_3, \ldots, \chi_n$ so that

$$|\chi_1 - \chi_2| \le |\chi_1 - \chi_3| \le \dots \le |\chi_1 - \chi_n|.$$

Put $|\chi_1 - \chi_i| = h^{-\rho_i}$ $(i = 2, 3, \dots, n)$. Take an arbitrary, but fixed in what follows, $\varepsilon > 0$, put $m = [n/\varepsilon] + 1$, and define the integers r_i by the inequalities

$$\frac{r_i}{m} \le \rho_i < \frac{r_i + 1}{m} \qquad (i = 2, 3, \dots, k). \tag{4}$$

Then there exist no more than $c(n, \varepsilon)$ distinct systems $(r_2, r_3, \dots, r_n)$ generated by polynomials $P \in \mathfrak{P}_n(h)$. If there exist infinitely many polynomials P satisfying condition (4), then

$$\sum_{j=2}^{n} (j-1)\frac{r_j}{m} \le n - 1, \tag{5}$$

as follows from considering the discriminant $D(P)$ of the polynomial P in the form

$$D(P) = h^{2n-2} \prod_{1 \le i < j \le n} (\chi_i - \chi_j)^2.$$

Polynomials P having the same systems $(r_2, r_3, \dots, r_n)$ are assigned to the first or the second class depending on whether the following conditions are fulfilled:

1) $\dfrac{r_2}{m} < \dfrac{n - s_1}{2}$;
2) $\dfrac{r_2}{m} \ge \dfrac{n - s_1}{2}$,

where $s_1 = \frac{1}{m}(r_3 + r_4 + \dots + r_n)$. Polynomials of the first class are considered in the same way as in the preceding note. In the case of polynomials of the second class, however, by virtue of (5), we have $r_2/m + 2s_1 \le n - 1$. Therefore

$$\frac{n - s_1}{2} + 2s_1 \le \frac{r_2}{m} + 2s_1 \le n - 1, \qquad s_1 \le \frac{n-2}{3}.$$

Further, $2r_3/m + s_1 \le 3s_1 \le n - 2$, so that $r_3/m \le (n - s_1)/2 - 1$. Consequently,

$$\frac{r_2}{m} \ge \frac{n - s_1}{2} > \frac{r_3}{m}. \tag{6}$$

Now let $\mathfrak{P}_n^k$ be the set of polynomials $P \in \bigcup_{h=1}^{\infty} \mathfrak{P}_n(h)$ satisfying the condition $2^{k-1} < h(P) \le 2^k = H$, where $h(P)$ is the height of P. Suppose that there exists a pair of polynomials $P_1, P_2 \in \mathfrak{P}_n^k$ satisfying $|\chi_1^{(1)} - \chi_1^{(2)}| < cH^{-(n-s_1)/2}$, where $\chi_1^{(1)}$, $\chi_1^{(2)}$ are, respectively, the roots of the polynomials P_1, P_2 relative to which are defi-

the membership of these polynomials in the second class has been determined. Then

$$\left|\chi_i^{(1)} - \chi_j^{(2)}\right| \ll 2H^{-\frac{1}{m} r_{\max(i,j)}} + cH^{-(n-s_1)/2}.$$

In view of (6), we now find

$$\left|\chi_i^{(1)} - \chi_j^{(2)}\right| \ll \begin{cases} cH^{-(n-s_1)/2}, & \text{if } \max(i,j) \leqslant 2, \\ H^{-\frac{1}{m} r_{\max(i,j)}}, & \text{if } \max(i,j) \geqslant 3. \end{cases}$$

Therefore

$$1 \leqslant |R(P_1, P_2)| \leqslant (h_1 h_2)^n \prod_{1 \leqslant i,j \leqslant n} \left|\chi_i^{(1)} - \chi_j^{(2)}\right|$$
$$\ll H^{2n} c^4 H^{-4(n-s_1)/2} \prod_{\max(i,j) \geqslant 3} H^{-\frac{1}{m} r_{\max(i,j)}} \leqslant c^4 H^{2s_1 - 5s_1} \leqslant c^4.$$

For sufficiently small $c > 0$, the inequality obtained is impossible. Consequently, the number of polynomials P of the second class in $\mathfrak{M}_n^k$ will be $\ll H^{(n-s_1)/2}$. This allows us to conclude that

$$w_n \leqslant \max(w_{n-1} + 1,\ n - 1) \qquad (n = 3, 4, \ldots).$$

Since $w_2 = 2$ and $w_n \geqslant n$, it must be that $w_n = n$ $(n = 2, 3, \ldots)$.

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REFERENCES

1. V. G. Sprindzhuk, DAN, **154**, No. 4 (1964).

Note: Figure translations are in progress. See original paper for figures.