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1964

SovietRxiv

Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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THE PROBLEM OF DIFFRACTION OF A PLANE WAVE BY A WEDGE MOVING WITH SUPERSONIC VELOCITY

(Presented by Academician L. I. Sedov on 16 XII 1963)

Propagating through a gas and increasing its pressure by ε, a weak plane compression wave at the instant $t = 0$ has reached the vertex of a wedge of finite aperture angle β_1, moving toward it with supersonic velocity, with an attached compression shock. The angle ϑ between the front of this wave and the axis of the wedge may be arbitrary. The typical pattern of the self-similar motion of the gas arising for $t > 0$ is shown in Fig. 1. In the region between the wedge, the shock, and the arcs of the Mach circles CD and AGF of radius $a_2 t$ (a is the speed of sound in this region), whose center coincides with the particle of gas that at $t = 0$ was at the vertex of the wedge, the self-similar diffraction under consideration develops. We associate the coordinate system x, y with the undisturbed gas between the wedge and the oblique compression shock, placing its origin at the center E of the indicated Mach circle and directing the x-axis perpendicular to the shock front. Denote by M_0 and a_0, respectively, the Mach number and the speed of sound of the flow incident on the wedge, and by α_1 the angle of inclination of the oblique compression shock to the axis of the wedge. This shock may then be regarded as a plane shock wave propagating along the x-axis with velocity $U = M_0 a_0 \sin\alpha_1$ and producing behind its front a gas velocity $V = [2U/(\chi + 1)](1 - a_0^2/U^2)$ (χ is the adiabatic exponent).

Fig. 1

The equations of plane unsteady motion of the gas, after their linearization with respect to the parameter ε, upon passing to self-similar coordinates $x = \bar{x}/a_2 t,\ y = \bar{y}/a_2 t$ and the Chaplygin transformation $r = 2R/(1 + R^2)$ (where $x = r\cos\theta,\ y = r\sin\theta$), are reduced to Laplace' s equation for the pressure disturbance. The only portion of the boundary of the disturbed region that is deformed under this transformation—the chord ABC—passes into the circle $2R\cos\theta = m(1 + R^2)$, orthogonal to the unit Mach circle. The boundary condition on this portion, corresponding to a weakly curved shock front, may be written in the form ([3]):

$$\frac{\partial p}{\partial n}\Big/\frac{\partial p}{\partial s} = (A\,\mathrm{tg}\,\theta - B\,\mathrm{ctg}\,\theta)/\sqrt{1-m^2\sec^2\theta}, \qquad \frac{\partial v}{\partial y} = \frac{B}{y}\frac{\partial p}{\partial y}, \tag{1}$$

where n and s are coordinates along the normal and tangent to the shock front, v and p are the y-component of the disturbance velocity and the pressure disturbance, $m = M_1 k_1/\sqrt{k_1^2+1}$, $k_1 = \mathrm{tg}(\alpha_1 - \beta_1)$, and M_1 is the Mach number of the flow along the surface of the wedge in the coordinate system attached to the wedge; the expressions for the constants A and B have the form

$$A = (U-V)^2\big[(\chi-1)(U-V)(2U-V)+a_2^2\big]^{-1} \tag{2}$$

$$B = U\big[(U-V)^2 - a_2^2\big](U-V)^{-1}\big[(U-V)(2U-V)(\chi-1)-a_2^2\big]^{-1}.$$

On the wall of the wedge we have the condition [3,4] $\partial p/\partial n = 0$, on the arc DC, $p = p_1$, on the arc AG, $p = p_2$, and on the arc GF, $p = 2p_2$. The quantities p_1 and p_2, as well as v, can be determined respectively with the aid of the relations in an oblique shock wave (see, for example, [5]) and with the aid of the results of a numerical calculation of the regular interaction of plane shock waves [6]. The function

$$z = \sigma + i\tau = \ln[(\lambda-\lambda_2)/(\lambda-\lambda_1)] - i[(\pi/2)-\alpha] - l,$$

$$\alpha = \arcsin(1/M_1), \qquad l = \frac{1}{4}\left[(1+k_1\sqrt{M_1^2-1})/(1-k_1\sqrt{M_1^2-1})\right] \tag{3}$$

$$\lambda_1 = e^{i(\alpha-\alpha_1)}, \qquad \lambda_2 = -e^{-i(\alpha+\alpha_1)},$$

maps our domain conformally onto the rectangle $-l < \sigma < l,\ 0 < \tau < \pi$. The condition on its right vertical side, corresponding to the shock front, then takes the form

$$\frac{\partial p}{\partial s}\Big/\frac{\partial p}{\partial n} = \frac{k_1 m_0^2 M_1\sqrt{M_1^2-1}\,(m_0 - M_1\cos\tau)\sin\tau}{Am_0^2(m_0-M_1\cos\tau)^2 - Bk_1^2M_1^2(M_1-m_0\cos\tau)^2} = b(\tau), \tag{4}$$

where $m_0 = \sqrt{1-k_1^2(M_1^2-1)}$. On the remaining part of the boundary the conditions do not change. The conformal mapping of the upper half-plane $\omega = \varphi + i\psi$ onto our rectangle is realized with the correspondence of points $z = 0,\ \omega = 0;\ z = i\pi,\ \omega = \infty;\ z = l,\ \omega = 1$; in this case the point $\omega = 1/k$ goes over into the point $z = l + i\pi$ (here k is the modulus of the elliptic integral). Along the right vertical side of the rectangle we then have the dependence

$$\tau = C \int_1^{\varphi} \left[(\varphi^2 - 1)(1 - k^2\varphi^2)\right]^{-1/2} d\varphi. \tag{5}$$

Substituting (5) into (4) (its formal procedure is not required), we obtain boundary conditions for the half-plane which, owing to the fact that $\partial p/\partial n = \partial p/\partial \psi$ and $\partial p/\partial s = \partial p/\partial \varphi$, may be interpreted as a boundary condition of the Riemann–Hilbert problem with discontinuous coefficients for the function $\Gamma(\omega)$:

$$a(\varphi)\frac{\partial p}{\partial \varphi} + b(\varphi)\frac{\partial p}{\partial \psi} = c(\varphi), \qquad \Gamma(\omega) = \frac{\partial p}{\partial \varphi} + i\frac{\partial p}{\partial \psi}, \tag{6}$$

where $a = 0$ for $-1/k < \varphi < -1$ and $a = 1$ on the remaining part of the real axis; $b = 0$ everywhere except the intervals $-1/k < \varphi < -1,\ 1 < \varphi < 1/k$, on the first of which $b = 1$, while on the second it is determined from (4) and (5); $c = -p_2\delta(\varphi - \varphi_0)$. The coordinates of the point G, corresponding to $\varphi = \varphi_0$, will be

$$\theta_0 = -(\alpha_1 + \vartheta),$$

$$\sigma_0 = [1 + \cos(\vartheta + \beta_1 - \alpha)]/[1 - \cos(\vartheta + \beta_1 + \alpha)] - l, \qquad \tau = 0.$$

Introduce the function $\Gamma_1(\omega)$ (see (1)): $\Gamma(\omega) = \Gamma_1(\omega)\Omega(\omega)$, where, as is shown in [7] for the analogous case,

$$\Omega(\omega) = [(\varphi + 1)(k\varphi + 1)]^{-1/2}.$$

For $\Gamma_1(\omega)$ we obtain a problem with continuous coefficients (which we denote by the same letters): $a \equiv 1;\ b = 0$ for $\varphi < 1,\ \varphi > 1/k$, and $b = b[\tau(\varphi)]$ according to (4) and (5) for $1 < \varphi < 1/k;\ c = -p_2\Omega^{-1}(\varphi_0)\delta(\varphi - \varphi_0)$.

The solution of the corresponding homogeneous problem, owing to the fact that, as is not difficult to establish, the index of the function $a + bi$ is zero for our contour, can be written in the form [1]

$$\Gamma_0 = i\beta \exp\left\{\frac{1}{\pi}\int_1^{1/k} \operatorname{arc\,tg} b[\tau(\xi)]\frac{d\xi}{\xi - \omega}\right\} = i\beta \exp\left[\frac{1}{\pi}F(\omega)\right]; \tag{7}$$

here ζ is the variable of integration along the φ-axis; β is, for the time being, an undetermined constant. The nonhomogeneous problem with discontinuous coefficients will then have the solution ([1,2])

$$\Gamma(\omega) = \Omega(\omega)\Gamma_1(\omega) = \Omega(\omega)\Gamma_0(\omega)\,[1 + L/(\varphi_0 - \omega)],$$

$$L = -p_2/\pi|\Omega(\varphi_0)|\Gamma_0(\varphi_0). \tag{8}$$

Integrating, we obtain the solution for the pressure in complex form:

$$p = \mathrm{Im}\left\{-\beta\int_{-1}^{\omega}|\Omega(\omega)|\exp\left[i\arg\Omega(\omega)+(1/\pi)F(\omega)\right]\left[1+L/(\varphi_0-\omega)\right]d\omega\right\}+p_2. \tag{9}$$

To determine the constant β we use the second relation (1): the integral along the curved portion of the shock front from its right-hand part must be equal to the difference of the y-components of the perturbed velocity on the two sides of this portion. From (7) and (8) it is clear that we need to know the limiting values, on the open contour, of the singular integral (7)—the interval $(1, 1/k)$. By the Sokhotskii formulas for the interior limiting values, we have

$$F^+ = -\frac{1}{C}\int_0^{\pi}\frac{\operatorname{arc\,tg} b(h) - \operatorname{arc\,tg} b(\tau)}{\zeta-\varphi}\sqrt{(\zeta^2-1)(1-k^2\zeta^2)}\,dh +$$
$$+\operatorname{arc\,tg} b(\tau)\left[\ln\left|\frac{1-k\varphi}{k(\varphi-1)}\right|+i\pi\right]. \tag{10}$$

Here h denotes the variable of integration along the τ-axis. Substituting (10) into the expression for $\partial p/\partial\varphi$, and the latter into the second relation (1), expressing y in it through τ, and integrating, we obtain a relation for determining the constant β:

$$\beta\frac{B\sqrt{k^2+1}}{m_0C}\int_0^{\pi}\frac{(M_1-m_0\cos\tau)\sqrt{(\varphi^2-1)(1-k^2\varphi^2)}}{m_0-M_1\cos\tau}\times$$
$$\times|\Omega|\exp\left(\frac{1}{\pi}\,\mathrm{Re}\,F^+\right)\frac{b(\tau)}{\sqrt{b^2+1}}\left(1+\frac{L}{\varphi_0-\varphi}\right)d\tau = v_2 - v_1;$$

the required dependence $\varphi(\tau)$ used here will be given below.

Separating the real and imaginary parts in expression (9), we obtain a formula for calculating the pressure field in the diffraction region:

$$p = \beta\int_0^{\psi}X\left[\frac{(\varphi_0-\varphi)\cos Y-\psi\sin Y}{(\varphi_0-\varphi)^2+\psi^2}\,L+\cos Y\right]d\psi +$$
$$+\beta\int_{-1}^{\varphi}X\left[\frac{(\varphi_0-\varphi)\sin Y+\psi\cos Y}{(\varphi_0-\varphi)^2+\psi^2}\,L+\sin Y\right]d\varphi.$$

The quantities X and Y entering here are defined as follows:

$$X = -|\Omega(\omega)| \exp\left[\frac{1}{\pi C}\int_0^\pi \frac{\operatorname{arc\,tg} b(h)(\zeta-\varphi)\sqrt{(\zeta^2-1)(1-k^2\zeta^2)}}{(\zeta-\varphi)^2+\psi^2}\,dh\right],$$

$$Y = \arg\Omega(\omega) + \frac{\psi}{\pi C}\int_0^\pi \frac{\operatorname{arc\,tg} b(h)\sqrt{(\zeta^2-1)(1-k^2\zeta^2)}}{(\zeta-\varphi)^2+\psi^2}\,dh.$$

Adopting the notation $\operatorname{sn}(\sigma/C,k)=\eta(\sigma)$, $\operatorname{sn}(\tau/C,k')=\eta'(\tau)$, we write the real and imaginary parts of $\omega=\operatorname{sn}(z/C,k)$ ([8]):

$$\varphi = \eta\frac{\sqrt{1-k'^2\eta a'^2}}{1+(k^2\eta^2-1)\eta a'^2}, \qquad \psi = \eta'\frac{\sqrt{(1-\eta a'^2)(1-\eta^2)(1-k^2\eta^2)}}{1+(k^2\eta^2-1)\eta a'^2}.$$

Finally, the expressions for R and ϑ in terms of σ and τ have the form:

$$\operatorname{tg}\vartheta = \frac{\operatorname{ch}(\sigma+l)-k_1\sqrt{M_1^2-1}\operatorname{sh}(\sigma+l)-M_1\cos\tau}{\sqrt{M_1^2-1}\operatorname{sh}(\sigma+l)+k_1\operatorname{ch}(\sigma+l)-k_1M_1\cos\tau},$$

$$R = \sqrt{\frac{\operatorname{ch}(\sigma+l)-\sin(a_1+\tau)}{\operatorname{ch}(\sigma+l)-\sin(a_1-\tau)}}.$$

Received
4 V 1963

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Note: Figure translations are in progress. See original paper for figures.