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Abstract

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MATHEMATICS

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ON THE STRUCTURAL DETERMINABILITY OF A LOCALLY NILPOTENT TORSION-FREE GROUP

(Presented by Academician A. I. Mal' tsev on 29 X 1963)

In the present paper we consider the connections between structural isomorphisms (projections) and one-to-one mappings of the elements of locally nilpotent torsion-free groups (pure groups). The basic concepts of the theory of nilpotent groups and of the theory of structural isomorphisms are assumed to be known (see ([1]-[4])). The investigations ([5], [6]) have proved very significant in the theory of projections of nilpotent groups: Ore proved that the subgroup lattice is distributive if and only if the group itself is locally cyclic; Baer established that every structural isomorphism of an abelian group of rank $r \geq 2$ is induced by two of its group isomorphisms. Beaumont showed ([7]) that Baer' s result loses force for abelian groups of rank $r = 1$.

Thus there arose the more general problem of the determinability of a locally nilpotent group by the structure of its subgroups and of the connection of its projections with one-to-one mappings of its elements. For locally nilpotent torsion-free groups this problem was posed in the survey ([8]). The works of many authors are devoted to the study of this problem (for a detailed bibliography see ([4], [9])). I rely on some facts obtained in these investigations.

Main theorem. *A nonabelian locally nilpotent torsion-free group G, containing at least two independent elements, is determined by the structure of its subgroups. Every structurally isomorphic mapping φ of the group G onto some group G^φ is induced by exactly one isomorphism and one anti-isomorphism between G and G^φ.*

The proof of the main result is based on a known local theorem ([10], [11]) and on a theorem which, in formulation, coincides with the main one, but in which the group G is assumed to be an arbitrary nonabelian finitely generated n-step nilpotent torsion-free group. The proof of this latter theorem on the inducibility of every projection of the group G by exactly one isomorphism and one anti-isomorphism is carried out by induction on the nilpotency class n of the group. It uses Baer' s result ([6]) for abelian groups ($n = 1$) and the case of metabelian groups ($n = 2$), which requires special consideration, as well as a number of facts

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concerning the structure of nilpotent groups containing independent elements of infinite order (recall that elements u, v are called independent if their cyclic groups $\{u\}, \{v\}$ have only the identity element in common).

We shall state some of the propositions that are used in the proof of the main theorem.

Definition. The series

$$G = G_1 \supset I(G_2) \supset \cdots \supset I(G_n) \supset e,$$

composed of the isolators of the terms of the lower central series

$$G = G_1 \supset G_2 \supset \cdots \supset G_n \supset e$$

of the n-nilpotent group G, will be called the **isolated lower central series** of this group.

Theorem 1. The isolator $I(G_m)$ of the m-th term $(m = 2, 3, \dots, n)$ of the lower central series of a pure n-nilpotent group G coincides with the intersection $\bigcap_\alpha H_\alpha$ of all isolated normal divisors H_α of the group G, the quotient groups G/H_α by which are torsion-free and $(m-1)$-nilpotent.

Theorem 2. Every projection φ matches the isolated lower central series of an n-nilpotent group G with that of its structurally isomorphic image G^φ.

Theorem 3. A metabelian group generated by two independent elements of infinite order is a free metabelian group if and only if no powers of these elements commute with each other.

Lemma 1. If an n-nilpotent group G contains at least two independent elements of infinite order, then in the quotient group G/G_n of this group by the penultimate term G_n of its lower central series there are at least two such elements.

Theorem 4. Suppose that a finitely generated n-nilpotent group contains at least two independent elements of infinite order. Then every maximal m-nilpotent subgroup of it $(m \leq n)$ also contains at least two independent elements of infinite order.

Lemma 2. Every projection φ of a finitely generated n-nilpotent group G, containing at least two independent elements of infinite order, is induced by two one-to-one correspondences φ_1 and φ_2 between the elements of the groups G and G^φ. The correspondences φ_1 and φ_2 are isomorphisms on each pair of Abelian subgroups A of G and A^φ of G^φ.

In what follows, when one of the two possible correspondences φ_1, φ_2 of Lemma 2 is established between the elements of the groups G and G^φ, it is denoted by

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the same symbol φ as the projection under consideration. The element of G^φ corresponding under φ to the element g of G is denoted by $\varphi(g) = g'$.

Theorem 5. Suppose that a finitely generated metabelian group G contains at least two independent elements of infinite order and that every pair of such elements generates in it a free metabelian subgroup. Then every projection φ carries the group G onto a group G^φ isomorphic to it, and is induced by exactly one isomorphism and one anti-isomorphism between G and G^φ.

Lemma 3. Suppose that a finitely generated n-nilpotent group G contains at least two independent elements of infinite order and that the commutator of any pair of such elements also has infinite order, provided the elements themselves generate in G a metabelian subgroup. Then on every metabelian (Abelian) subgroup A of G, one of the correspondences defined by Lemma 2 is an isomorphism, and the other is an anti-isomorphism (respectively, the second isomorphism).

Lemma 4. Suppose that the conditions of Lemma 3 are satisfied, and suppose that the subgroup $(G_n)^\varphi$ under the projection φ is the image of the penultimate term G_n of the lower central series of the group G. Then the correspondences φ_1 and φ_2, defined by Lemma 2, induce on the quotient groups G/G_n and $G^\varphi/(G_n)^\varphi$ natural one-to-one correspondences $\overline{\varphi}_1$ and $\overline{\varphi}_2$, which are isomorphisms on each pair of Abelian subgroups $\overline{M}$ and $\overline{M}^\varphi$ of these quotient groups. The correspondences $\overline{\varphi}_1$ and $\overline{\varphi}_2$ induce the natural structural isomorphism $\overline{\varphi}$ between G/G_n and $G^\varphi/(G_n)^\varphi$, generated by the projection φ.

Lemma 5. Let G be a pure finitely generated n-nilpotent group with at least two independent elements, and let φ be a projection carrying G onto the group G^φ. Then the one-to-one correspondences φ_1 and φ_2 between the elements of G and G^φ defined by Lemma 2 have the following properties:

If a and b are any two elements of G, then the equalities

$$\varphi_1(a) = a', \qquad \varphi_1(b) = b', \tag{1}$$

where a', b' are elements of G', imply for the products $ab, a'b'$ and the commutators $(a, b), (a', b')$ of these elements the relations

$$\varphi_1(ab) = a'b'c_1', \qquad \varphi((a, b)) = (a', b')c_2'. \tag{2}$$

At the same time, from the equalities

$$\varphi_2(a) = {a'}^{-1}, \qquad \varphi_2(b) = {b'}^{-1} \tag{3}$$

it follows that

$$\varphi_2(ab) = {b'}^{-1}{a'}^{-1}c_3', \qquad \varphi_2((a, b)) = ({b'}^{-1}, {a'}^{-1})c_4', \tag{4}$$

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where c_i' $(i = 1, \dots, 4)$ are certain elements of $(G_n)^\varphi$.

The formulations of the following lemmas refer only to one of the two possible correspondences φ_1, φ_2 between the elements of the groups G and G^φ, namely to the correspondence $\varphi = \varphi_1$, which becomes an isomorphism $\bar{\varphi}$ between the factor groups G/G_n and $G^\varphi/(G_n)^\varphi$. It is easy to imagine how the corresponding assertions sound when $\bar{\varphi}$ becomes an anti-isomorphism between G/G_n and $G^\varphi/(G_n)^\varphi$.

Lemma 6. Under the hypotheses of Lemma 5 and for the correspondence φ chosen in accordance with (1)–(2), the following relations hold:

$$\varphi(a^k) = {a'}^k, \qquad \varphi(b^l) = {b'}^l \quad (k, l \text{ arbitrary integers}),$$

$$\varphi(a^\varepsilon b^\mu) = {a'}^\varepsilon {b'}^\mu c_5, \qquad \varphi((a^\varepsilon, b^\mu)) = ({a'}^\varepsilon, {b'}^\mu) c_6,$$

$$|\varepsilon| = |\mu| = 1, \qquad c_5, c_6 \in (G_n)^\varphi.$$

Lemma 7. Under the hypotheses of Lemma 5, the correspondence φ, chosen in accordance with (1)–(2), assigns to each commutator $k = (x_1, \dots, x_m)$ of weight $m \geq 3$ from the m-th term G_m of the lower central series of the group G, written with respect to the first powers of the elements a and b, the commutator

$$k' = \varphi(k) = (x_1', \dots, x_m'), \qquad x_i' = \varphi(x_i) \quad (i = 1, 2, \dots, m)$$

from the image $(G_m)^\varphi$ of the subgroup G_m.

Lemma 8. Consider any set $k_1, \dots, k_s$ of simple commutators, written in terms of the first powers of the elements a, b of the group G. Suppose the weight of each of them is not less than 3. Then, under the hypotheses of Lemma 5, the correspondence φ, chosen in accordance with (1)–(2), gives

$$\varphi(k_1^{\alpha_1} \cdots k_s^{\alpha_s}) = {k'}_1^{\alpha_1} \cdots k_{s'}^{\alpha_s}, \qquad \varphi(k_i) = k_i' \quad (i = 1, 2, \dots, s).$$

Lemma 9. Every projectivity φ of a pure finitely generated n-nilpotent group G, containing at least two independent elements, maps its lower central series to the lower central series of the group G^φ.

Lemma 10. Suppose the hypotheses of Lemma 5 are fulfilled and the correspondence φ is chosen in accordance with (1)–(2). Then:

1°. For any integer λ,

$$\varphi(a^\lambda b^\lambda) = {a'}^\lambda {b'}^\lambda c_1^\lambda c_2^{-\lambda(\lambda-1)/2}.$$

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2°. The elements c_i $(i = 1, \dots, 4)$ in relations (1)–(4) turn out to be identity elements.

In the proof of some of the lemmas listed above, a number of properties of commutators of elements are used. Many of them are considered in papers ([12–14]) and in the book ([3]); others are established directly.

Let now φ be some projection of a pure finitely generated n-nilpotent group G, containing at least two independent elements, onto the group G^φ. In accordance with Lemma 2, we establish correspondences φ_1 and φ_2 between the elements of G and G^φ that induce the projection φ under consideration; we fix one of them and denote it by the same symbol φ as the projection itself. If a and b are any two elements of G, then from the preceding lemmas it follows that either $\varphi(a, b) = \varphi(a)\varphi(b)$, or $\varphi(ab) = \varphi(b)\varphi(a)$. In this case, as shown in [15], for all pairs of elements from G and G^φ either only the first or only the second alternative is realized. This means that the correspondence φ is either an isomorphism or an anti-isomorphism between G and G^φ.

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