ON THE INVERSE PROBLEM FOR THE STURM-LIOUVILLE EQUATION

M. G. GASYMOV

1964

SovietRxiv

Abstract

Full Text

ON THE INVERSE PROBLEM FOR THE STURM-LIOUVILLE EQUATION

M. G. GASYMOV

(Presented by Academician I. G. Petrovskii, 5 VIII 1963)

Mathematics

1. Consider the differential equation

$$-y'' + q(x)y = sy \qquad (0 \leq x < \infty) \tag{1}$$

with boundary condition

$$y'(0) - hy(0) = 0. \tag{2}$$

Here $q(x)$ is a real locally summable function of x; h is a real number. Let the function $q(x)$ be such that every self-adjoint extension of equation (1) has a discrete spectrum. Denote by $\{\lambda_n(h)\}$ the eigenvalues of the boundary-value problem (1)–(2) (if necessary, we prescribe one more boundary condition at infinity), numbered in increasing order, and by $\{\varphi_n(x,h)\}$ the corresponding orthonormal eigenfunctions. Setting $h = h_1$ and $h = h_2$, where $h_1 \neq h_2$, we obtain two different boundary-value problems with eigenvalues $\{\lambda_n(h_1)\}$ and $\{\lambda_n(h_2)\}$. It is known (^{1}) that equation (1) is determined uniquely by two spectra $\{\lambda_n(h_1)\}$ and $\{\lambda_n(h_2)\}$, but it is not known when sequences of alternating numbers $\{\lambda_n(h_1)\}$ and $\{\lambda_n(h_2)\}$ are eigenvalues of one and the same equation of type (1).

The present paper is devoted to the solution of this problem for one class of operators. In the case of the regular Sturm-Liouville equation, a similar problem was studied by M. G. Krein (^{2,3}) and B. M. Levitan (^{4}).

2. Thus, consider the boundary-value problem (1)–(2) for $h = h_1$ and $h = h_2$, where $h_1 \neq h_2$. It is not difficult to show that

$$\int \varphi_n(x,h_1)\varphi_n(x,h_2)\,dx\,[\lambda_n(h_2) - \lambda_n(h_1)] = (h_2 - h_1)\varphi_n(0,h_1)\varphi_n(0,h_2). \tag{3}$$

Therefore, as $h_1 \to h_2 = h$, we obtain

$$\lambda_n'(h) = \varphi_n^2(0, h).$$

Hence follows the formula*

$$\lambda_n(h_2) - \lambda_n(h_1) = \int_{h_1}^{h_2} \varphi_n^2(0, h)\, dh. \tag{4}$$

We denote the spectral function of the problem (1)–(2) by $\rho_h(\lambda)$ and introduce the functions

$$\sigma_i(\lambda) = \frac{1}{h_2 - h_1} \sum_{\lambda_n(h_i)<\lambda} \{\lambda_n(h_2) - \lambda_n(h_1)\}, \qquad i = 1, 2. \tag{5}$$

---

* Using the asymptotic behavior, as $\lambda \to \infty$, of the spectral function (^{7})

$$\rho_h(\lambda) = \sum_{\lambda_n(h)<\lambda} \varphi_n^2(0, h) = \frac{2}{\pi}\sqrt{\lambda} - h + o(1)$$

for the problem (1)–(2), and formula (4), one can more simply derive formulas (11) and (12) of (^{5}).

Formula (4) gives grounds to suppose that the functions $\sigma_i(\lambda)$, $i = 1, 2$, possess all the properties of the spectral function of the Sturm-Liouville operator. We shall prove this hypothesis for one class of potentials $q(x)$. First we define this class of potentials.

Definition 1. We say that $q(x)$ belongs to the class $\Omega_{1/4}$ if, for some $h = h_1$, the following conditions are satisfied: a) $\rho_{h_1}(\lambda) = \frac{2}{\pi}\sqrt{\lambda} - h_1 + O\left(\frac{1}{\lambda^{1/4+\delta}}\right)$, where $\delta > 0$; b) $\lim\limits_{h\to\infty} \frac{\rho_{h_1}(\lambda_n) - \rho_{h_1}(\lambda_{n-1})}{\sqrt{\lambda_n} - \sqrt{\lambda_{n-1}}} = \frac{2}{\pi}$; c) $\lambda_{n+1}(h_1) - \lambda_n(h_1) \geq \alpha$, where $\alpha > 0$ is a positive constant.

It can be shown that if, for some $h = h_1$, conditions a), b), and c) are satisfied, then they are also satisfied for other values of h (generally speaking, δ and α depend on h).

The class of functions $\Omega_{1/4}$ is nonempty. The function $q(x) = \beta x^2 + p(x)$, where $\beta > 0$ and $p(x)$ is a finite summable function, belongs to the class $\Omega_{1/4}$.

For simplicity in what follows we shall assume that the spectrum of problem (1)–(2) is bounded below. (The case in which the spectrum is not bounded below is more complicated only technically.) Then one may assume that $\lambda_n(h_1) > 0$ and $\lambda_n(h_2) > 0$ for $n = 0, 1, ...$ We formulate the main result of the present paper.

Theorem 1. *In order that the sequences of interlacing numbers* $\{\lambda_n(h_1)\}_{n=0}^{\infty}$ *and* $\{\lambda_n(h_2)\}_{n=0}^{\infty}$ *be the eigenvalues of one and the same equation of type (1) with locally summable potential* $q(x) \in \Omega_{1/4}$*, but with different boundary conditions at zero, it is necessary and sufficient that the functions* $\sigma_i(\lambda)$*,* $i = 1, 2$*, defined by formula (5), be the spectral functions of certain singular equations of type (1) with locally summable potentials from the class* $\Omega_{1/4}$*.*

We indicate the lines of the proof.

Necessity. To prove necessity, it is first necessary to prove (see [6]) that the functions

$$F_i(x) = \int_0^{\infty} \frac{1 - \cos\sqrt{\lambda}\,x}{\lambda}\, d\sigma_i(\lambda), \qquad i = 1, 2, \tag{6}$$

have third summable derivatives. We shall give the further explanation of the proof for $h_1 = 0$ and $i = 1$ (the general case differs only by technical complexity). Denote by $\varphi(x, s)$ and $\theta(x, s)$ the solutions of equation (1) with the following initial conditions: $\varphi(0, s) = 1$, $\varphi'(0, s) = 0$ and $\theta(0, s) = 0$, $\theta'(0, s) = 1$. Further, let $f(x, s) \in L_2(0, \infty)$ and $f(x, s) = \theta(x, s) + m(s)\varphi(x, s)$. Then, obviously, the poles of the function $m(s)$ coincide with the eigenvalues of problem (1)–(2) for $h = 0$. It is also not difficult to show that the poles of the function $m_{h_2}(s) = \dfrac{m(s)}{1 - h_2 m(s)}$ coincide with the eigenvalues of problem (1)–(2) for $h = h_2$. Then, as is easily seen,

$$\sigma_1(\lambda) = \rho_0(\lambda) + \rho(\lambda), \tag{7}$$

where $\rho_0(\lambda)$ is the spectral function of problem (1)–(2) for $h = 0$, and

$$\rho(\lambda) = \frac{1}{2\pi i h_2} \sum_{n=2}^{\infty} \frac{h_2^n}{n} \oint_{|s|=\lambda} m^n(s)\, ds. \tag{8}$$

It is now obvious that the function $F_1(x)$ has a third locally summable derivative if the function

$$F(x) = \int_0^{\infty} \frac{1 - \cos\sqrt{\lambda}\,x}{\lambda}\, d\rho(\lambda)$$

has a third locally summable derivative. Consequently, we have to prove that $F'''(x)$ is a locally summable function. To do this one must use the asymptotic formula as $\lambda \to \infty$

$$\rho(\lambda) = -\frac{h_2}{2} + \frac{2}{3\pi}\frac{h_2^2}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda^{3/4+\delta_1}}\right), \tag{9}$$

where $\delta_1 > 0$. This formula is obtained from the following important lemma.

Lemma 1. The function $m(s)$ for potentials of the class $\Omega_{1/4}$, as $|s| \to \infty$, behaves as follows:

a) outside the domain $R\{|\operatorname{Im} s| \le 1,\ \operatorname{Re} s \ge 0\}$

$$m(s) = -\frac{i}{\sqrt{s}} + O\left[\frac{1}{|s|^{1/4+\delta}(|s|\theta(-u) + |v|)}\right], \tag{10}$$

where $s = u + iv$ and

$$\theta(u) = \begin{cases} 0, & \text{for } u < 0, \\ 1, & \text{for } u > 0; \end{cases}$$

b) for $|\operatorname{Im} s| \le 1$ and $\operatorname{Re} s = \dfrac{\lambda_{n-1} + \lambda_n}{2}$

$$|m(s)| = O\left(\frac{\ln \lambda_n}{\sqrt{\lambda_n}}\right). \tag{11}$$

Further, from our arguments it follows that the function $\sigma_1(\lambda)$ possesses all the properties of the spectral function of the Sturm-Liouville operator, and from it one can reconstruct the locally summable function $q_1(x)$. From formula (9) it follows that $q_1(x) \in \Omega_{1/4}$. Thus, the necessity of the theorem is proved.

Sufficiency. To prove sufficiency, let us note that from two spectra $\{\lambda_n(h_1)\}$ and $\{\lambda_n(h_2)\}$ the spectral function of problem (1)–(2) for $h = h_1$ is determined by the following formulas*:

$$\rho_{h_1}(\lambda) = \sum_{\lambda_n(h_1) < \lambda} \beta_n, \tag{12}$$

$$\beta_n = \frac{\lambda_n(h_2) - \lambda_n(h_1)}{h_2 - h_1} \prod_{k=0}^{\infty}{}' \left(1 + \frac{\lambda_k(h_2) - \lambda_k(h_1)}{\lambda_k(h_1) - \lambda_n(h_1)}\right), \tag{13}$$

where the sign $\prod'$ means that in the infinite product the term with $k = n$ is absent. (Formulas (12) and (13) can be obtained with the aid of the results of the paper (^{5}).) Therefore, if we are given sequences of interlacing numbers $\{\lambda_n(h_1)\}$ and $\{\lambda_n(h_2)\}$ which satisfy the conditions of Theorem 1, then we can construct from them the function $\rho_{h_1}(\lambda)$ by means of formulas (12) and (13) (formula (11) of the paper (^{5}) makes it possible, from the numbers $\{\lambda_n(h_1)\}$ and $\{\lambda_n(h_2)\}$, to find the numbers h_1 and h_2). Now it is necessary to prove that $\rho_{h_1}(\lambda)$ is the spectral function of a boundary-value problem of type (1)–(2) for $h = h_1$ with potential $q(x) \in \Omega_{1/4}$. To this end let us note that

$$\rho_{h_1}(\lambda) = -\frac{1}{2\pi i}\frac{1}{h_2 - h_1}\oint_{|s|=\lambda} \exp\left[\sum_{k=0}^{\infty} \ln\left(1 + \frac{\lambda_k(h_2) - \lambda_k(h_1)}{\lambda_k(h_1) - s}\right)\right] ds. \tag{14}$$

Expanding the function $\ln\left(1 + \frac{\lambda_k(h_2) - \lambda_k(h_1)}{\lambda_k(h_1) - s}\right)$ in a series in powers of $\frac{\lambda_k(h_2) - \lambda_k(h_1)}{\lambda_k(h_1) - s}$ and using the expansion of the exponential function and some estimates for func-

* Formula (13) was found by B. M. Levitan jointly with the author.

estimates of

$$\sum_{k=0}^{\infty} \frac{\lambda_k(h_2) - \lambda_k(h_1)}{\lambda_k(h_1) - s},$$

which are similar to estimates (9) and (10), one can show that

$$\rho_{h_1}(\lambda) = \sigma_1(\lambda) + \frac{h_2 - h_1}{2} + \frac{(h_2 - h_1)^2 - 3(h_2^2 - h_1^2)}{3\pi\sqrt{\lambda}} + O\left(\frac{1}{\lambda^{3/4+\delta_2}}\right), \tag{15}$$

where $\delta_2 > 0$. From this formula it is clear that the function $\rho_{h_1}(\lambda)$ has all the properties of the spectral function of a Sturm–Liouville operator, and from it one can reconstruct the potential $q(x) \in \Omega_{1/4}$. It remains only to prove that the eigenvalues $\{\mu_n(h_2)\}$ of the reconstructed equation with the boundary condition $y'(0) - h_2 y(0) = 0$ coincide with the numbers $\{\lambda_n(h_2)\}$, but we shall not do this here.

3. In this section we formulate one result concerning the regular Sturm–Liouville equation. Consider the equation

$$-y'' + q(x)y = sy, \qquad 0 \le x \le \pi, \tag{16}$$

and the boundary conditions

$$y'(0) - h_1 y(0) = 0, \tag{17}$$

$$y'(0) - h_2 y(0) = 0, \tag{18}$$

$$y'(\pi) + Hy(\pi) = 0. \tag{19}$$

Here $q(x)$ is a summable function; h_1, h_2, H are real numbers, with $h_1 \neq h_2$. Denote by $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ the eigenvalues of the boundary-value problems (16)–(17)–(19) and (16)–(18)–(19), respectively. With the aid of the methods used in the present work, one can prove the following theorem.

Theorem 2. In order that the sequences of interlacing numbers $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ be the eigenvalues of one and the same Sturm–Liouville equation of type (16) with summable potential $q(x)$, but with different boundary conditions at zero and identical boundary conditions at the point π, it is necessary and sufficient that $\lambda_n = n^2 + a_0 + o(1)$, $\mu_n = n^2 + b_0 + o(1)$, where $a_0 \neq b_0$, and that the function

$$F(x) = \sum_{n=0}^{\infty} \left\{ \frac{\mu_n - \lambda_n}{h_2 - h_1} \cos \sqrt{\lambda_n}\, x - \frac{2}{\pi} \cos nx \right\} \qquad (0 \leq x \leq 2\pi), \qquad (20)$$

where $h_2 - h_1 = \dfrac{\pi}{2}(b_0 - a_0)$, have a summable derivative.

The author expresses his sincere gratitude to Prof. B. M. Levitan for his help in carrying out the work.

Received
20 V 1963

REFERENCES

1. V. A. Marchenko, Tr. Mosk. matem. obshch., **1**, 327 (1952).
2. M. G. Krein, DAN, **76**, No. 1, 21 (1951).
3. M. G. Krein, DAN, **94**, No. 5, 767 (1954).
4. B. M. Levitan, DAN, **150**, No. 3 (1963).
5. M. G. Gasymov, B. M. Levitan, DAN, **151**, No. 5 (1963).
6. I. M. Gel'fand, B. M. Levitan, Izv. AN SSSR, ser. matem., **15**, 309 (1951).
7. B. M. Levitan, Izv. AN SSSR, ser. matem., **19**, 381 (1955).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.