O. N. KATSKOVA, P. I. CHUSHKIN

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Abstract

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O. N. KATSKOVA, P. I. CHUSHKIN

ON ONE SCHEME OF THE NUMERICAL METHOD OF CHARACTERISTICS

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For the numerical calculation of a supersonic steady gas flow around three-dimensional bodies in the region between the shock wave and the surface of the body, one may apply the three-dimensional method of characteristics or a finite-difference method. However, the practical realization of these methods on electronic computers requires the creation of very complex programs. In this connection, one numerical method appears highly promising, the idea of which is as follows.

Let us use the variables x, $\xi = (r - r_{\mathrm{t}})/(r_{\mathrm{w}} - r_{\mathrm{t}})$, φ, where x, r, φ are cylindrical coordinates; $r = r_{\mathrm{t}}(x, \varphi)$ is the prescribed equation of the body; $r = r_{\mathrm{w}}(x, \varphi)$ is the sought equation of the shock wave. Considering a system of equally spaced meridional planes $\varphi = \mathrm{const}$, we shall approximate, in the three-dimensional equations of the problem, the functions by trigonometric polynomials in φ with interpolation nodes on these planes. Then the original system of three-dimensional equations is reduced to a system of two-dimensional equations in x and ξ, relating the values of the functions on the individual meridional planes. To solve this hyperbolic system we shall apply the two-dimensional method of characteristics, in which the calculation is carried out by layers bounded by the planes $x = \mathrm{const}$, and in which the characteristics on each new layer pass through selected points with values $\xi = \mathrm{const}$.*

In order to test such a scheme of the numerical method of characteristics by layers, in the present note we shall apply it to a particular case—the calculation of supersonic nonisentropic axisymmetric flow of a perfect gas around a body of revolution with a contour without breaks. For convenience of machine calculations, as in ([2,3]), we take as the principal functions $\beta = \sqrt{\mathrm{M}^2 - 1}$, $\zeta = \tan\theta$, the entropy $S = \ln(p/\rho^{\chi})$, and the radius r, where M is the Mach number; θ is the angle of inclination of the velocity vector to the flow axis; ρ is the dimensionless density, referred to the density of the oncoming flow ρ_∞; p is the dimensionless pressure, referred to $\rho_\infty a_{\mathrm{cr}}^2$ (a_{cr} is the critical speed of sound); χ is the adiabatic exponent. Then, in the axisymmetric case under consideration, in the variables x, ξ, the differential equations and compatibility relations for the characteristics of the first and second families, as well as the differential equation of the streamlines, will have, respectively, the form

$$\frac{d\xi}{dx} = \frac{1}{\delta}\left(\frac{\beta\zeta + 1}{\beta - \zeta} - \lambda\right) \equiv A_1, \qquad d\zeta + K\,d\beta + L\,dx - P\,dS = 0, \tag{1}$$

$$\frac{d\xi}{dx} = \frac{1}{\delta}\left(\frac{\beta\zeta - 1}{\beta + \zeta} - \lambda\right) \equiv A_2, \qquad d\zeta - J\,d\beta - N\,dx + Q\,dS = 0, \tag{2}$$

$$\frac{d\xi}{dx} = \frac{1}{\delta}(\zeta - \lambda) \equiv A_3, \tag{3}$$

where

$$\delta = r\ - r\ , \qquad \lambda = \left(\frac{dr}{dx} - \frac{dr}{dx}\right)\xi + \frac{dr}{dx},$$

$$J = K = -\frac{2\beta^2(\zeta^2 + 1)}{(\chi + 1)(\beta^2 + 1)(\varepsilon\beta^2 + 1)}, \qquad L = \frac{\zeta(\zeta^2 + 1)}{r(\beta - \zeta)},$$

$$N = \frac{\zeta(\zeta^2 + 1)}{r(\beta + \zeta)}, \qquad P = Q = \frac{\beta(\zeta^2 + 1)}{\chi(\chi - 1)(\beta^2 + 1)}, \qquad \varepsilon = \frac{\chi - 1}{\chi + 1},$$

and

$$r_\xi = \delta\xi + r\ . \tag{4}$$

We shall now describe the computational scheme. Suppose that on some layer $x = x_0$, located between the shock wave ($\xi = 1$) and the body ($\xi = 0$) in the supersonic region of the flow, the values of the basic functions are known at a number of points n, with the corresponding $\xi = \xi_n = \text{const}$. We shall find the values of these functions on the next layer $x_0 + \Delta x$.

The calculation begins with the determination of point 3 on the shock wave (Fig. 1a), and is carried out here by selecting the value $\tau = dr\ /dx$, the tangent of the angle of inclination of the shock wave to the x-axis. Assigning a value τ_3 close to the value of τ at point $n + 1$ on the shock wave on the layer x_0, we compute, from the relations on the shock wave, the quantities

Fig. 1

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* In work (1) a scheme of the numerical method of characteristics by layers is considered, but applied to the calculation of one-dimensional unsteady gas flows; moreover, it does not require fulfillment of the last condition. A scheme of the method of characteristics by layers, but only for the case of isentropic one-dimensional gas flows, is also given in (5).

Fig. 1

Figure 1: Fig. 1

$$\rho_3 = \frac{w_\infty^2 \tau_3^2}{1 - \varepsilon w_\infty^2 + \tau_3^2}, \qquad \zeta_3 = \frac{\tau_3(\rho_3 - 1)}{\tau_3^2 + \rho_3} \left(w_\infty^2 = \frac{(\chi+1)M_\infty^2}{2 + (\chi - 1)M_\infty^2} \right),$$

$$\beta_3 = \sqrt{\frac{w_3^2 - 1}{1 - \varepsilon w_3^2}}, \qquad w_3^2 = \frac{1}{1 + \tau_3^2} \left[w_\infty^2 + \frac{1}{\rho_3}(1 - \varepsilon w_\infty^2 + \tau_3^2) \right],$$

$$S_3 = \ln \left[\frac{2}{\chi+1} \frac{w_\infty^2 \tau_3^2}{1 + \tau_3^2} - \frac{\chi - 1}{2\chi}(1 - \varepsilon w_\infty^2) \right] - \chi \ln \rho_3,$$

$$r_3 = r \ = r_{n-1} + \frac{\Delta x}{2}(\tau_{n-1} + \tau_3).$$

Next, drawing from point 3 a characteristic of the first family I, we write equations (1) in finite-difference form:

$$\xi_1 = \xi_3 - A_1 \Delta x \qquad (\xi_3 = 1), \tag{5}$$

$$\zeta_3 - \zeta_1 + K(\beta_3 - \beta_1) + L\Delta x - P(S_3 - S_1) = 0. \tag{6}$$

Formula (5), in which A_1 is computed from the quantities at point 3, gives ξ_1. By quadratic interpolation to this value ξ_1 from the points $n-1$, n, and $n+1$ on the known layer x_0, we determine β_1, ζ_1, and S_1, and from formula (4) we find r_1. Then, by satisfying equality (6), the correctness of the choice of τ_3 is checked. The subsequent approximations, in which the values of A_1 in (5) averaged over points 1 and 3 are now taken, make it possible to determine τ_3 with the required accuracy. After the selection is completed, one can compute the generalized stream function Ψ, which is related to the ordinary stream function ψ by the relation $d\Psi = (\chi e^S)^{1/(\chi-1)} d\psi$. On the shock wave at point 3 we have

$$\Psi \ = \Psi_{n-1} + \chi^{\frac{1}{\chi-1}} \frac{w_\infty}{4} \left(\exp \frac{S_1}{\chi - 1} + \exp \frac{S_3}{\chi - 1} \right) (r_3^2 - r_{n-1}^2). \tag{7}$$

After computing the point on the shock wave, we proceed to the determination of the functions at an interior point 3 of the flow field, corresponding to some prescribed $\xi_3 = \xi_n$ (Fig. 1b). From point 3 we draw the characteristics of the first family I and of the second family II, and the streamline III. Writing equations (1)–(3) in finite-difference form, we obtain, for determining the unknown quantities β_3, ξ_3, and S_3, a system that must be solved by iteration; in the first

Fig. 2

Figure 2: Fig. 2

iteration the values of the functions at point 3 are taken to be the same as at point n on the layer x_0. First ξ_1, ξ_2, and ξ_4 are computed from the formulas

Fig. 2

$$\xi_i = \xi_3 - A_i \Delta x \qquad (i = 1,\, 2,\, 4), \tag{8}$$

where in the first iteration A_i are determined from point n. With the aid of quadratic interpolation to these values ξ_i at points $n-1$, n, and $n+1$, one finds β_1, β_2, ξ_i, and S_i $(i = 1,\, 2,\, 4)$, and from formula (4), r_1 and r_2. The refined values of the sought functions at point 3 are obtained from the expressions

$$\beta_3 = \frac{1}{K+J}\{\xi_1 + K\beta_1 - (L+N)\Delta x - P(S_3 - S_1) - \xi_2 + J\beta_2 + Q(S_3 - S_2)\},$$

$$\xi_3 = J(\beta_3 - \beta_2) + N\Delta x - Q(S_3 - S_2), \qquad S_3 = S_4,$$

where the coefficients are taken averaged over points 1 and 3 or 2 and 3. Subsequent iterations are computed analogously, but in them the coefficients A_i are taken averaged. Correct signs are usually obtained after three iterations.

When computing point 3 on the body contour ($\xi_3 = 0$), it is necessary to determine only the quantity β_3, since the quantities $r_3 = r_t$, $\zeta_3 = dr_t/dx$, and S_3 are known here. Draw the characteristic of the second family II from point 3 (Fig. 1b). Taking in the first iteration $\beta_3 = \beta_{n+1}$, we find from formula (8), for $i = 2$, the quantity ξ_2, and by quadratic interpolation over the points $n-1$, n, and $n+1$ we obtain β_2, ζ_2, and S_2, and from expression (4), r_2. The refined value β_3 is determined as follows:

$$\beta_3 = \beta_2[\xi_3 - \xi_2 - N\Delta x + Q(S_3 - S_2)].$$

Subsequent iterations are carried out analogously. From the final value of β_3, the pressure on the body is found:

$$p = e^{-S/(\varkappa-1)}[\varkappa(\varepsilon\beta^2 + 1)]^{-\varkappa/(\varkappa-1)},$$

and by integrating p along the body contour the drag coefficient c_x is obtained.

After the functions have been computed on the entire layer, the value of the stream function Ψ_t on the body is computed from them:

$$\Psi_t = \Psi_{\mathrm{B}} - \delta \int_0^1 r \sqrt{\frac{\beta^2 + 1}{(\varepsilon\beta^2 + 1)^{1/\varepsilon}}}\, d\xi,$$

where Ψ_{B} is given by equality (7). The obtained Ψ_t is compared with its known value on the body, which serves as a check on the accuracy of the entire computation. The step size Δx must be related to the step $\Delta\xi$ and is chosen from considerations of accuracy and stability of the computations.

The developed scheme of the numerical method of characteristics by layers was tested on the example of computing a supersonic flow region past blunt bodies placed in an air stream ($\chi = 1.4$) with Mach number $M_\infty = \infty$. The cases considered were a cone with semi-vertex angle $\omega = 5^\circ$ and a cylinder $\omega = 0^\circ$ with spherical blunt noses. The flow was computed from the known initial data (4) in the section $x/R = 1$, where R is the radius of the blunt nose (the origin of coordinates is placed at the front point of the body). The results of the computations are given in Fig. 2, where the pressure distribution over the body (solid line) and the shape of the shock wave (dashed line) are plotted. On the same graphs, the corresponding computational data (4), obtained by the ordinary method of characteristics, are plotted as points. Comparison of the results shows that the considered scheme of the method of characteristics by layers gives good accuracy. Let us note that the same accuracy is also obtained when comparing the flow fields. We also point out that the scheme presented is readily generalized to the case of a real gas in a state of thermodynamic equilibrium.

Computing Center
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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