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# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

G. A. FREIMAN

# ON THE ADDITION OF FINITE SETS

(Presented by Academician P. S. Novikov on May 6, 1964)

**Notation.**  $K = \{a_0, a_1, \dots, a_{k-1}\}$ ;  $a_i$  are integers;  $a_i < a_{i+1}$ ,  $i = 0, 1, \dots, k-2$ ;  $K + K = 2K$  is the set of numbers of the form  $a_i + a_j$ , in which identical sums are taken only once;  $T(M)$  is the number of elements of the finite set  $M$ ;  $T = T(2K)$ ;  $Z_n$  is the additive group of integer vectors of the Euclidean space  $E_n$ ;  $A$  is an increasing sequence of nonnegative integers  $a_0 = 1, a_1, \dots, a_i, \dots$ ;  $A(x), A_2(x)$  are the numbers of terms of the sequence  $A$  (respectively  $2A$ ) not exceeding  $x$ ;

$$\delta(A) = \lim_{x \rightarrow \infty} \frac{A(x)}{x}$$

is the asymptotic density of the sequence  $A$ ;  $D$  is a set of points in  $E_n$ ;  $\mu^*(D)$  is the outer measure of the set  $D$ .

The following results describe the structure of the finite set  $K$ , the sequence  $A$ , and the set  $D$  in the case when, respectively,  $T(2K) < Ck$ ,  $\delta(2A) < C\delta(A)$ , and  $\mu^*(2D) < C\mu^*(D)$ , where  $C$  is an arbitrary positive constant.

**Definition 1.** Subsets  $B'$  and  $C'$  of sets  $B$  and  $C$  with an algebraic operation are called **isomorphic up to the  $s$ -th degree** if the mappings  $B^i \rightarrow C^i$ ,  $2 \leq i \leq s+1$ , naturally induced by a one-to-one mapping  $B' \rightarrow C'$ , are one-to-one.

**Theorem 1.** Let  $T < Ck$ ,  $C \geq 2$ ,  $k \geq k_0$ , where  $k_0$  is a sufficiently large positive number. There exist a homomorphism  $\varphi : Z_n \rightarrow Z_1$  and a convex set  $D \subset E_n$  such that:

- 1)  $K \subset (D \cap Z_n)\varphi$ ;
- 2)  $D \cap Z_n$  and  $(D \cap Z_n)\varphi$  are isomorphic up to the first degree;
- 3)  $T(D \cap Z_n) < c_1k$ ,  $c_1 = c_1(C) > 0$ ;
- 4)  $n \leq [C - 1]$ .

**Definition 2.** The parallelepiped  $\sigma_1\mathbf{x}_1 + \sigma_2\mathbf{x}_2 + \dots + \sigma_n\mathbf{x}_n$ ,  $0 \leq \sigma_i < 1$ ,  $1 \leq i \leq n$ , is called **canonical** if

$$(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{22} & \dots & x_{2n} & \\ 0 & \dots & x_{nn} & \end{pmatrix}, \quad (1)$$

where  $x_{ij}$  are natural numbers and  $|x_{ii}| < x_{jj}$ ,  $i < j$ ,  $1 \leq j \leq n$ .

For the proof of Theorem 1 we shall need the following two lemmas.

**Lemma 1.** *Let  $T(D \cap Z_n) = V$ , where  $D$  is a convex set in  $E_n$ . There exist a homomorphism  $\varphi_1 : Z_m \rightarrow Z_n$ , a canonical parallelepiped  $H \subset Z_m$ , and a vector  $\mathbf{w} \in Z_n$  such that: 1)  $D \cap Z_n \subset (H \cap Z_m)\varphi_1 + \mathbf{w}$ , 2)  $V(H) < c_2 V$ ,  $c_2 = c_2(n) > 0$ .*

**Proof.** Let  $L$  be a plane of minimal dimension  $r$  containing  $D \cap Z_n$  ( $L$  may coincide with  $E_n$ ). Then  $L \cap D = D_1$  is a convex set. There is an integral frame of dimension  $r$  lying in  $D_1$ , i.e. there exist vectors  $\mathbf{u}, \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r$  such that  $\mathbf{u}, \mathbf{u} + \mathbf{w}_1, \mathbf{u} + \mathbf{w}_2, \dots, \mathbf{u} + \mathbf{w}_r \subset L$ , and the vectors  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r$  are linearly independent. Let  $\mathbf{y}_1 = \mathbf{x}_1, \mathbf{y}_i$  be an integer point on  $\mathbf{u} + L(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_i)$ , whose distance from  $\mathbf{u} + L(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{i-1})$  is positive and minimal ( $2 \leq i \leq r$ ).

Define the nondegenerate linear transformation  $\varphi_2 : E_r \rightarrow E_n$  by the relations  $e_i \varphi_2 = y_i$ ,  $1 \leq i \leq r$ . Let  $D_2 = (D_1 - \mathbf{u})\varphi_2^{-1}$ . Applying Lemma 4 from (5), we obtain a parallelepiped  $H_1 \supset D_2$ , whose edges can be represented in the form (1),  $x_{ij} \geq 1$ . Since a convex body containing only one integral point has volume not exceeding  $2^n$  (Minkowski's theorem, (1), p. 184), we have  $V(H_1) < c_3 V(D_2) = c_3 V(D_3) \leq 2^n c_3 V$ .

There exist a canonical parallelepiped  $H$ , a homomorphism  $\varphi_3 : E_r \rightarrow E_r$ , and an integral point  $v$ , for which

$$H_1 \cap Z_r \subset (H \cap Z_r)\varphi_3 + v, \quad V(H) \leq 2^r V(H_1).$$

Then  $\varphi_1 = \varphi_3^{-1}\varphi_2$ ,  $w = u + v\varphi_2$ .

**Lemma 2.** *Let  $\bar{K} \subset Z_n$ ,  $T(2\bar{K}) < Ck$ ,  $C \geq 2$ ,  $\bar{K} \subset D \subset E_n$ , where  $D$  is a convex set for which  $T(D \cap Z_n) = V$ . There exist a homomorphism  $\varphi_4 : Z_m \rightarrow Z_n$ , a convex set  $D_1 \subset E_m$ , and a set  $K' \subset \bar{K}$ , for which*

$$T(K') > \varepsilon k, \quad \varepsilon > 0,$$

such that: 1)  $K' \subset (D_1 \cap Z_m)\varphi_4$ ; 2)

$$T(D_1 \cap Z_m) < c_4 V(V/k)^{-c_5}, \quad c_4 = c_4(n, C), \quad c_5 = c_5(n, C).$$

The arguments needed in order to show that Theorem 1 follows from Lemma 2 are carried out essentially in (5).

**Proof of Theorem 1.** From Lemmas 1 and 2 it follows that there exist a homomorphism  $\varphi_5 : Z_m \rightarrow Z_n$ , a canonical parallelepiped  $H \subset E_m$ , and a vector  $w \in Z_n$  such that: 1)  $K' \subset (H \cap Z_m)\varphi_5 + w$ ; 2)

$$V(H) < c_6 V(V/k)^{-c_5}.$$

As shown at the end of the proof of Lemma 3 in <sup>(5)</sup>, one can specify a homomorphism  $\varphi_6 : Z_p \rightarrow Z_n$ , a canonical parallelepiped  $H_1 \subset E_p$ , and  $w_1 \in E_n$  such that: 1)  $\bar{K} \subset (H_1 \cap Z_p)\varphi_6 + w_1$ , 2)

$$V(H_1) < c_7 V(V/k)^{-c_5}.$$

The proof is completed in the same way as the proof of Theorem 1 in <sup>(5)</sup>, with the sole difference that, in constructing the sequence  $\{\varphi_i, H_i, w_i\}$  indicated there, instead of Lemma 3 one uses the just-described construction of  $\varphi_6, H_1$ , and  $w_1$ , carried out with the aid of Lemma 2.

For the proof of Lemma 2 we shall need the following.

**Lemma 3.** *Let  $r$  random variables  $\xi_1, \xi_2, \dots, \xi_r$  be given, in general dependent, each of which assumes the two values 0 and 1. If*

$$P(\xi_i = 0) = \sum_{\substack{u_j=0,1 \\ j \neq i \\ u_i=0}} P_{u_1 u_2 \dots u_r} \geq \gamma > \frac{1}{2}, \quad 1 \leq i \leq r,$$

where  $P_{u_1 u_2 \dots u_r} = P(\xi_1 = u_1, \xi_2 = u_2, \dots, \xi_r = u_r)$ , then there exists a set of values  $u_1, u_2, \dots, u_r$  such that

$$P_{u_1 u_2 \dots u_r} \geq c_8 V^{-r} c_9^r, \quad \text{where } c_8 = c_8(\gamma), \quad c_3 = (1 - \gamma)^{1-\gamma} \gamma^\gamma > \frac{1}{2}.$$

**Proof.** Let

$$m = \max_{u_i} P_{u_1 u_2 \dots u_r}.$$

Then

$$\gamma r \leq \sum_{i=1}^r P(\xi_i = 0) = \sum_{s=1}^r s P_{u_1 u_2 \dots u_r} < s_0 m + (s_0 - 1) C_r^1 m + \dots + 1 \cdot C_r^{s_0 - 1} m + r - s_0,$$

$$+ s_0, \quad \text{where } s_0 = [(1 - \gamma)r + 2] \text{ and } m > 1/F, \quad F = C_r^{s_0 - 1} + 2C_r^{s_0 - 2} + \dots + s_0.$$

From

$$C_r^{s_0 - 1} < c_{10}(\gamma)/V^{-r} c_9^r$$

(see, for example, <sup>(2)</sup>, p. 245, formula (12)) the validity of the lemma follows.

**Proof of Lemma 2.** In view of Lemma 1 one may assume that  $D \subset H - w$ , where  $H$  is a canonical parallelepiped of volume  $V$ ,  $w \in Z_n$ .

We retain the notation  $W, S, J_1, Q_i, H_a, J'_1, h_i$  of the work <sup>(4)</sup>. Change the quantity  $Q_i$ , setting it equal to  $Q_i = h_i/M$ , where  $M = (V/k)^{1/4n}$ . Define the set  $J'_1$  as follows:

$$J'_1 = \bigcup_{p_i, q_i} H_a, \quad \text{where } (p_i, q_i) = 1, \quad 1 \leq i \leq n, \quad 1 \leq q_i \leq q_0,$$

$q_0 = M^2$ . Then

$$\text{mes } J'_1 \ll \sum \frac{2^n}{q_1 q_2 \dots q_n Q_1 Q_2 \dots Q_n} \ll \frac{2^n}{V} M^{3n}.$$

Suppose that for some  $\bar{a} \in \bar{J}'_1$ ,  $\bar{a} \in H_a$ , the inequality

$$|S(\bar{a})| > \frac{1}{\sqrt{C + \varepsilon/2}} k, \quad \varepsilon > 0$$

holds. Since

$$|(\mathbf{z}, \mathbf{x})| \leq \sum_{i=1}^n |(\mathbf{z}, \mathbf{x}_i)| < c_{11} \sum_{i=1}^n \frac{h_i}{q_i Q^i} < c_{11} n \frac{1}{M},$$

we have

$$|S(\bar{a}) - S(\mathbf{a})| \leq \sum_{x \in K} |e^{2\pi i(\mathbf{z}, \mathbf{x})} - 1| \leq c_{12} \frac{k}{M},$$

whence, for  $k$  sufficiently large, the inequality follows

$$|S(\mathbf{a})| \geq \frac{1}{\sqrt{C + \varepsilon}} k. \quad (2)$$

We shall show that one can choose  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r \in \bar{J}'_1$  in such a way that, for any choice of  $n + r$  integers  $A_j$ ,  $1 \leq j \leq n + r$ , for which

$$|A_j| \leq R, \quad R = (V/k)^{1/3(n+r)}, \quad \sum_{i=n+1}^{n+r} A_j^2 > 0,$$

for at least one  $i$ ,  $1 \leq i \leq n$ , the inequality

$$\left| A_1 x_{1i} + A_2 x_{2i} + \dots + A_{n x_{ni}} - (\mathbf{a}_1, \mathbf{x}_i) A_{n+1} - \dots - (\mathbf{a}_r, \mathbf{x}_i) A_{n+r} \right| \leq g = (V/k)^{1/3n} \quad (3)$$

would fail.

Suppose the points  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{j-1}$  have already been chosen. Put in (3)  $A_i = 0$ ,  $i \geq n + j + 1$ ,  $A_{n+j} = 1$ . Then

$$G_{ji} - g \leq (\mathbf{a}_j, \mathbf{x}_i) \leq G_{ji} + g, \quad (4)$$

where

$$G_{ji} = \pm (A_1 x_{1i} + A_2 x_{2i} + \dots + A_{n x_{ni}} - (\mathbf{a}_1, \mathbf{x}_i) A_{n+1} - \dots - (\mathbf{a}_{j-1}, \mathbf{x}_i) A_{n+j-1}).$$

Condition (4) defines a strip perpendicular to  $\mathbf{x}_i$ , of width  $2g/|\mathbf{x}_i|$ ; the collection of conditions (4) for all  $i$  is a parallelepiped of volume not exceeding  $c_{13} g^n / V$ .

All possible combinations of the values  $A_i$ ,  $1 \leq i \leq n + j - 1$ , determine  $(2R + 1)^{n+j-1}$  parallelepipeds, whose total volume does not exceed

$$c_{14} \frac{g^n}{V} R^{n+j-1}.$$

The sum  $U_j$  of the parallelepipeds  $H_a$  having common points with those constructed has volume not exceeding

$$c_{15} \frac{g^n}{V} R^{n+j-1}.$$

Since

$$\text{mes } J'_1 + r c_{15} \frac{g^n}{V} R^{n+r-1} < \frac{1}{2} \text{mes } J_1,$$

the choice of  $\mathbf{a}_j \notin U_j$ ,  $1 \leq j \leq r$ , is possible for any finite  $r$ .

From inequality (2), with the aid of Lemma 1, it follows from (3) that for  $k_1$  vectors  $\mathbf{x}$  from  $\bar{K}$ ,

$$\left( k_1 > \frac{1 + 1/\sqrt{C + \varepsilon}}{2} k \right),$$

the conditions

$$\{(\mathbf{a}, \mathbf{x})\} = A_x + \beta + \theta_x, \quad 0 \leq \theta_x < \frac{1}{2},$$

are satisfied, where  $A_x$  is an integer,  $\beta = P/Q$ ,  $Q = [q_1, q_2, \dots, q_n]$ . Hence, for  $\mathbf{a}_i$ ,  $1 \leq i \leq r$ , we obtain

$$\begin{aligned} \frac{p_{i1}}{q_{i1}} Q_i x_1 + \frac{p_{i2}}{q_{i2}} Q_i x_2 + \dots + \frac{p_{in}}{q_{in}} Q_i x_n + Q_i x_{n+i} &= P_i + s, \\ 0 \leq s < \frac{P}{2}, \quad 1 \leq i \leq r. \end{aligned} \quad (5)$$

This condition is satisfied for  $k_{1i}$  points of  $\bar{K}$ , while for the remaining  $k_{2i} = k - k_{1i}$  points condition (5) holds with  $P/2 \leq s < P$ .

Consider the hyperplanes  $L_{i0}, L_{i1}, L_{i2} \subset E_{n+r}$ , defined by relation (5) respectively with  $s = 0, P/2, P$ ;  $P_{i0}$  ( $P_{i1}$ ) is the strip between  $L_{i0}$  and  $L_{i1}$  ( $L_{i1}$  and  $L_{i2}$ ), including  $L_{i0}$  ( $L_{i1}$ );  $N$  is the cylinder bounded by the hyperplanes passing through each of the faces  $H \subset E_n = L(e_1, e_2, \dots, e_n)$  parallel to  $L(e_{n+1}, e_{n+2}, \dots, e_{n+r})$ ;

$$H_{u_1 u_2 \dots u_r} = \left( \bigcap_{i=1}^r P_{iu_i} \right) \cap N; \quad H' = \bigcup_{u_i=0, 1 \leq i \leq r} H_{u_1 u_2 \dots u_r}.$$

Define  $\varphi_7 : Z_{n+r} \rightarrow Z_n$  by the relation

$$(x_1, x_2, \dots, x_{n+r}) \varphi_7 = (x_1, x_2, \dots, x_n).$$

In view of Lemma 3, there exists  $H_{u_1 u_2, \dots, u_r} = H''$  such that

$$k'' T((H'' \cap Z_{n+r}) \varphi_\gamma \cap K) > c_{16} \sqrt{r} c_9^r k, \quad \gamma = \frac{1 + 1/\sqrt{C} + \varepsilon}{2}.$$

Since, for sufficiently large  $r$ ,

$$T(2(\overline{K} \varphi_\gamma^{-1} \cap H'')) < T(2\overline{K}) < Ck < 2^r k'',$$

it follows, in view of Lemma 2 from (5), that there exists a hyperplane  $L_1 \subset E_{n+r}$  such that

$$T(L_1 \cap (K \varphi_\gamma^{-1} \cap H'')) > \varepsilon k, \quad \varepsilon > 0, \quad \varepsilon = \varepsilon(n, C),$$

and  $L_1 \parallel L(e_1, e_2, \dots, e_n)$ . We may suppose that the set of points

$$L_1 \cap H' \cap Z_{n+r}$$

is not contained in a plane having dimension smaller than  $n + r - 1$ . Consider the set  $G$  of hyperplanes  $L$  parallel to  $L_1$  and passing through integral points.

The area of a fundamental parallelepiped of the lattice in the plane  $L_1$  is equal to  $1/h$ , where  $h$  is the distance between two neighboring hyperplanes  $Z \subset G$ . Minkowski's inequality for the product of the successive minima of a convex closed symmetric body of volume

$$V(V\lambda_1 \lambda_2 \cdots \lambda_n \leq 2^n, \text{ see (1), p. 187})$$

shows that in  $L_1$  there exists a fundamental parallelepiped whose edge lengths  $u_1, u_2, \dots, u_{n+r-1}$  do not exceed  $2^n/h^2$ .

The edges of the parallelepiped  $H'$  are equal to

$$y_i = x_i - (a_1, x_i)e_{n+1} - \cdots - (a_r, x_i)e_{n+r}, \quad 1 \leq i \leq n,$$

and to  $e_s$ ,  $n + 1 \leq s \leq n + r$ . If  $f = \max_i V_i$ , where  $V_i$  is the volume of the parallelepiped constructed on  $y_i$  and  $u_1, u_2, \dots, u_{n+r-1}$ , then there exist at least  $[f]$  planes  $Z \subset G$  whose intersections with  $H'$  are nonempty. But

$$V_i = \left\| \begin{array}{cccc} u_{11} & u_{12} & \cdots & u_{1n+r} \\ u_{21} & u_{22} & \cdots & u_{2n+r} \\ \cdot & \cdot & \cdot & \cdot \\ u_{n+r-11} & u_{n+r-12} & \cdots & u_{n+r-1n+r} \\ y_{1i} & y_{2i} & \cdots & y_{n+ri} \end{array} \right\| =$$

$$= |A_1 x_{1i} + A_2 x_{2i} + \cdots + A_{n x_{ni}} - (a_1 x_i) A_{n+1} - \cdots - (a^r, x_i) A_{n+r}|,$$

where  $|A_i| \leq c_{17} h$ , and it cannot be that

$$\sum_{j=n+1}^{n+r} A_j^2 = 0,$$

since  $L_1 \parallel [e_1, e_2, \dots, e_n]$ .

If

$$h^{-1} \geq c_{18}(V/k)^{c_5}, \quad c_5 = 1/6(n+r)^2,$$

then  $[f] \geq g$ . If, however,

$$h^{-1} < c_{18}(V/k)^{c_5},$$

then for the number  $p$  of nonempty intersections of  $H'$  with  $L \subset G$  the inequality

$$p > c_{19}(V/k)^{c_5}$$

holds, whence it follows that always

$$T(H' \cap L_1 \cap Z_{n+r}) < c_{20}V(V/k)^{-c_5}.$$

Lemma 2 is proved.

From Theorem 1 there immediately follows the following strengthening of the Brunn-Minkowski inequality:

**Theorem 2.** Let

$$\mu^*(2D) \leq C\mu^*(D), \quad C \geq 2^n.$$

There exist  $Z_m \rightarrow E_n$ , a rectangular canonical parallelepiped  $H$ , and a convex set  $D_1$  such that: 1)  $D \subset (H \cap Z_m)\varphi + D_1$ ; 2)  $(H \cap Z_m) \times D \subset E_m \times E_n$  is isomorphic to  $(H \cap Z_m)\varphi + D_1$ ; 3)  $T(H \cap Z_n)\mu^*(D_1) < c_{21}\mu^*(D)$ ,  $c_{21} = c_{21}(C, n)$ ; 4)  $m \leq [C - 2^n]$ .

From Theorem 1 follows the validity of Theorem 4 in (5) for any  $C \geq 1$ .

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