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Abstract

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INVESTIGATION OF THE NAVIER–STOKES EQUATIONS BY METHODS OF THE THEORY OF PARABOLIC EQUATIONS IN BANACH SPACES

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We consider the first boundary-value problem for the nonstationary nonlinear system

$$\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + v_k \frac{\partial \mathbf{v}}{\partial x_k} = \operatorname{grad} p + \mathbf{f}, \qquad \operatorname{div} \mathbf{v} = 0 \quad (0 < t \le T;\ x \in \Omega); \qquad (1)$$

$$\mathbf{v} = 0 \quad (0 < t \le T;\ x \in S); \qquad \mathbf{v}(0, x) = \mathbf{v}_0(x) \quad (x \in \overline{\Omega}).$$

Here Ω is an open bounded domain of m-dimensional space with boundary S, and $\overline{\Omega} = \Omega + S$; $\mathbf{v}(t,x) = (v_1(t,x), \dots, v_{\underline{m}}(t,x))$ and $p(t,x)$ are the unknown velocity and pressure, defined on $[0,T] \times \overline{\Omega}$ and satisfying (1); $\mathbf{f}(t,x)$ and $\mathbf{v}_0(x)$ are the prescribed force and initial velocity. If in equations (1) we discard the nonlinear terms $v_k \partial \mathbf{v}/\partial x_k$, then we obtain a linearized problem, which we shall call problem $(*)$.

Denote by H_q the closure in the metric $L_q(\Omega)$ $(1 < q < \infty)$ of the set of all smooth solenoidal vector-functions defined on $\overline{\Omega}$ and vanishing near S. In ([4]) an operator P of orthogonal projection in L_2 onto H_2 was constructed. It can be shown that it is a bounded projection operator also in L_q onto H_q. Projecting problems (1) and $(*)$ onto H_q, we arrive at the nonlinear

$$\frac{d\mathbf{v}}{dt} + \nu A \mathbf{v} + P\left(v_k \frac{\partial \mathbf{v}}{\partial x_k}\right) = P\mathbf{f}, \qquad \mathbf{v}(0) = \mathbf{v}_0 \quad (A = -P\Delta) \qquad (1')$$

and the corresponding linear $(*)$ Cauchy problems for ordinary differential equations of first order in the Banach space H_q. A solution of problem $(1')$ is declared to be a solution of problem (1).

In works ([1–4]) problem $(1')$ was studied in the Hilbert space H_2 in the case when $m = 2, 3$. Passing to a Banach space has made it possible to study problem $(1')$

for arbitrary m and to prove the existence of its unique solution for arbitrary $\mathbf{v}_0 \in H_m$. If the force $\mathbf{f}$ and the boundary S are sufficiently smooth, then by means of the methods developed in ([3]) it can be shown that this solution of problem (1′) will be a classical solution of problem (1).

1. In ([5]) estimates were established for solutions of problem (∗) or (∗′). If in these inequalities one sets

$$\mathbf{u}(t,x) = e^{\lambda t}\mathbf{u}_0(x), \qquad p(t,x) = e^{\lambda t} p_0(x), \qquad \mathbf{f}(t,x) = e^{\lambda t}\mathbf{f}_0(x),$$

where

$$\lambda \mathbf{u}_0 - \Delta \mathbf{u}_0 = \operatorname{grad} p_0 + \mathbf{f}_0, \qquad \operatorname{div} \mathbf{u}_0 = 0 \ (x \in \Omega); \qquad \mathbf{u}_0(x) = 0 \ (x \in S),$$

then we obtain that for any $q \in (1,\infty)$, sufficiently large $\sigma_0 = \sigma_0(q) > 0$, $\operatorname{Re}\lambda \geq \sigma_0$, and integer $l \geq 0$, the inequality

$$|\lambda| \, \|\mathbf{u}_0\|_{W_q^{2l}} + \|\mathbf{u}_0\|_{W_q^{2l+2}} + \|p_0\|_{W_q^{2l+1}} \leq c(q,l) \, \|P\mathbf{f}_0\|_{W_q^{2l}} \qquad (W_q^0 = L_q).$$

Hence it follows

Theorem 1. *For* $\operatorname{Re}\lambda \geqslant \sigma_0$ *the operator* $A + \lambda I$ *has an inverse in* H_q^{2l} *and*

$$\left\|(A+\lambda I)^{-1}\right\|_{H_q^{2l}\to H_q^{2l}} \leqslant c(q,l)(|\lambda|+1)^{-1}, \qquad \left\|(A+\lambda I)^{-1}\right\|_{H_q^{2l}\to H_q^{2l+2}} \leqslant c(q,l), \tag{2}$$

where $H_q^{2l} = H_q \cap W_q^{2l}$.*

The first of inequalities (2) means that the operator A generates in the spaces H_q^{2l} the analytic semigroup e^{-tA} ([8,9]), and therefore the theory of equations of parabolic type in Banach spaces ([10]) is applicable to the investigation of problem (1′).

2. From the results ([2,11]) it follows (cf. ([4]))

Lemma 1. *For arbitrary* $q \in (1,\infty)$, $(\alpha,\beta) \in G(m)$, $\mathbf{v} \in D[(-\Delta)^{\alpha}]$, $\mathbf{w} \in D[(-\Delta)^{\beta}]$ *(and* $\operatorname{div}\mathbf{v} = 0$, *if* $\alpha+\beta > m/2q$*) the inequality holds*

$$\left\|(-\Delta)^{\alpha+\beta-(m+q)/2q}\frac{\partial}{\partial x_k}(v_k w)\right\|_{L_q} \leqslant c(q,\alpha,\beta)\,\|(-\Delta)^{\alpha}\mathbf{v}\|_{L_q}\left\|(-\Delta)^{\beta}\mathbf{w}\right\|_{L_q}.$$

Here $G(m)$ is the set of points of the plane (α,β) lying on the triangle with vertices $(0,0)$, $((m+q)/4q,(m+q)/4q)$, and $(0,(m+q)/2q)$, except for the points $(0,0)$, $(0,(m+q)/2q)$, and (α,β) with $\beta \leqslant 1/2$ and $\alpha > m/2q - 1/2$; $-\Delta$ is the operator in L_q defined by the differential expression $-\partial^2/\partial x_i^2$ on $\mathring{W}_q^2$; $D[(-\Delta)^\gamma]$ is the domain of definition of the operator $(-\Delta)^\gamma$.

Obviously, without loss of generality one may assume that the first of inequalities (2) for $l=0$ is satisfied for $\operatorname{Re}\lambda \geqslant 0$. Then all inequalities (2) will also be satisfied for $\operatorname{Re}\lambda \geqslant 0$. Therefore fractional powers A^γ are defined (see ([9])), and the inequality

$$\left\|A^\gamma e^{-tA}\right\|_{H_q^{2l}\to H_q^{2l}} \leqslant c(\gamma,q,l)t^{-\gamma} \qquad (t>0). \tag{3}$$

holds.

An operator B, acting in a Banach space E and having an everywhere dense domain of definition $D(B)$, is called **weakly positive** if for every $\lambda \geqslant 0$ the operator $A+\lambda I$ has a bounded inverse whose norm satisfies the inequality

$$\left\|(A+\lambda I)^{-1}\right\|_{E\to E} \leqslant c(E)(\lambda+1)^{-1}.$$

The operator $-\Delta$ is weakly positive in L_q (see, for example, ([10])). As noted above, the operator A is weakly positive in H_q.

In ([12]) fractional powers of a weakly positive operator were defined. By the methods developed in ([13]), the following is proved.

Lemma 2. *Let B be weakly positive in E, $F \in (E \to E_1)$ and admit a closure, $D(F) \supset D(B)$, and for every $z \in D(B)$ and some $\alpha \in (0,1)$*

$$\|Fz\|_{E_1} \leqslant c(E,E_1)\|Bz\|_E^\alpha \|z\|_E^{1-\alpha}.$$

Then for arbitrary $0<\varepsilon \leqslant 1-\alpha$, $0<\eta \leqslant \alpha$ and $z \in D(B^{\alpha+\varepsilon})$

$$\|Fz\|_{E_1} \leqslant c(E,E_1,\varepsilon,\eta)\left\|B^{\alpha+\varepsilon}z\right\|_E^{\eta/(\varepsilon+\eta)}\left\|B^{\alpha-\eta}z\right\|_E^{\varepsilon/(\varepsilon+\eta)}.$$

For weakly positive operators the moment inequality ([12]) is valid.

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* In particular, we obtain the result of ([6]). We note that the estimates of problem $(*)$, obtained in ([7]), make it possible analogously to obtain estimates of the resolvent of A in Hölder norms.

Therefore

$$\|(-\Delta)^\alpha \mathbf{v}\|_{L_q} \le c(\alpha,q)\|(-\Delta)\mathbf{v}\|_{L_q}^\alpha \|\mathbf{v}\|_{L_q}^{1-\alpha} \quad (\mathbf{v} \in D[(-\Delta)]).$$

By virtue of (6)

$$\|(-\Delta)\mathbf{v}\|_{L_q} \le c\|A\mathbf{v}\|_{H_q}.$$

It then follows from Lemma 2 that

$$\|(-\Delta)^{\alpha}\mathbf{v}\|_{L_q} \le c(\alpha,\varepsilon,\eta,q)\|A^{\alpha+\varepsilon}\mathbf{v}\|_{H_q}^{\eta/(\varepsilon+\eta)}\|A^{\alpha-\eta}\mathbf{v}\|_{H_q}^{\varepsilon/(\varepsilon+\eta)}$$

$$(0<\alpha<1;\ 0<\varepsilon\le 1-\alpha;\ 0<\eta\le\alpha;\ \mathbf{v}\in D[A^{\alpha+\varepsilon}]).$$

Hence, and from (3), in turn it follows that for any $t>0$, $-1\le\alpha\le 1$, $-1\le\beta\le 1$, $\alpha+\beta>0$, the operators $(-\Delta)^{\alpha}e^{-tA}P(-\Delta)^{\beta}$ and $(-\Delta)^{\alpha}e^{-tA}A^{\beta}$ admit closures and

$$\|(-\Delta)^{\alpha}e^{-tA}P(-\Delta)^{\beta}\|_{L_q\to L_q},\quad \|(-\Delta)^{\alpha}e^{-tA}A^{\beta}\|_{H_q\to L_q}\le c(\alpha,\beta,q)t^{-(\alpha+\beta)}.$$

3. The results of the preceding item make it possible to study problem (1′) in any H_q, as was done in (4) in H_2. Separating in (1′) the linear part and setting $\mathbf{w}(t,\mu,\gamma)=t^{\mu-\gamma}(-\Delta)^{\mu}\mathbf{v}(t)$ $(\mu=\alpha,\beta;\ (\alpha,\beta)\in G(m);\ 1<q\le m;\ (m-q)/2q\le\gamma<\alpha)$, we obtain the system

$$\mathbf{w}(t,\mu,\gamma)=t^{\mu-\gamma}(-\Delta)^{\mu}e^{-tvA}\mathbf{v}_0+t^{\mu-\gamma}(-\Delta)^{\mu}\int_0^t e^{-(t-s)vA}P\mathbf{f}\,ds$$
$$-t^{\mu-\gamma}\int_0^t\{(-\Delta)^{\mu}e^{-(t-s)vA}P(-\Delta)^{[(m+q)/2q]-(\alpha+\beta)}\}\times$$
$$\times(-\Delta)^{\alpha+\beta-(m+q)/2q}\frac{\partial}{\partial x_k}\left[(-\Delta)^{-\alpha}w_k(s,\alpha,\gamma)(-\Delta)^{-\beta}\mathbf{w}(s,\beta,\gamma)\right]s^{2\gamma-(\alpha+\beta)}\,ds. \tag{4}$$

Denote by $\vec{\varphi}_i$ the i-th term on the right-hand side of (4). If $\mathbf{v}_0\in D(A^{\gamma})$, then from the estimates of item 2 it follows that $\vec{\varphi}_1$ is continuous on $[0,T]$ and $\vec{\varphi}_1\to 0$ as $t\to 0$. If $\mathbf{w}(t,\mu,\gamma)$ is continuous on $[0,T]$, then $\vec{\varphi}_3$ is continuous on $[0,T]$ and $\vec{\varphi}_3\to 0$ as $t\to 0$ (Lemma 1 and the corollary to Lemma 2).

Let $\mathbf{f}$ be such that $\vec{\varphi}_2$ is continuous on $[0,T]$ and $\vec{\varphi}_2\to 0$ as $t\to 0$. Put

$$M=M(T,\alpha,\beta,\gamma)=\max_{t,\mu}\|\vec{\varphi}_1+\vec{\varphi}_2\|_{L_q},$$

$$N=N(T,\alpha,\beta,\gamma)=$$

$$= \max_{t,\mu} t^{\mu-\gamma} \int_0^t \|(-\Delta)^\mu e^{-(t-s)vA} P(-\Delta)^{(m+q)/2q-(\alpha+\beta)}\|_{L_q \to L_q} s^{2\gamma-(\alpha+\beta)}\, ds\, c(\alpha,\beta,q)^*.$$

The following holds (cf. (4)).

Theorem 2. The system (4) has a unique solution in $C[0,T]$, if

$$MN < \frac{1}{4}, \tag{5}$$

and this solution can be found by the method of successive approximations.

Here $C[0,T]$ denotes the Banach space of continuous vector-functions $[\mathbf{w}_1(t), \mathbf{w}_2(t)]$ with values in $L_q \times L_q$. Condition (5) is always satisfied on a small interval $[0,T]$, i.e., a local existence theorem holds. Note that it has been proved for any $\mathbf{v}_0 \in D(A^{(m-q)/2q})$ $(1 < q \le m)$, i.e., for any $\mathbf{v}_0 \in H_m$.

* c is the same constant as in Lemma 1.

If (5) is satisfied for large $T > 0$, or for $T = \infty$, then a nonlocal existence theorem holds.

Theorem 3. Under the conditions of Theorem 2,

$$\|w(t,\mu,\gamma)\|_{L_q} \le M.$$

This theorem makes it possible to study the stability of the trivial solution with respect to perturbations of the initial velocities v_0 and forces f.

4. If one uses the known a priori estimate in L_2 for solutions of the Navier –Stokes equations, then one can obtain an existence theorem and give an estimate for solutions under weaker restrictions on v_0 and f (cf. ([4])).

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Note: Figure translations are in progress. See original paper for figures.