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Abstract

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ON A GENERAL MIXED PROBLEM

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The mixed problem for general two-dimensional hyperbolic equations has been the subject of numerous investigations by many mathematicians ($^{4-8}$). Either problems of a special type were considered (^{4,5,8}), or particular results were obtained for general problems (^{6,7}). The practical importance of such problems is beyond doubt (^{3}).

In the present note a general mixed problem is considered for a two-dimensional integro-differential equation of arbitrary order. As a special case, this problem contains the general mixed problem for a general two-dimensional hyperbolic equation. The question of the existence and uniqueness of a classical solution of the problem is studied.

1. Let, in the rectangle $\Pi\{0 \le t \le T;\ 0 \le x \le l\}$ in the (x,t)-plane, the integro-differential equation be given

$$\sum_{i=0}^{m}\sum_{j=0}^{i} a_{ij}(x,t)\frac{\partial^i u(x,t)}{\partial t^{i-j}\partial x^j} + \int_0^t \sum_{i=0}^{m-1}\sum_{j=0}^{i} b_{ij}(x,t,\tau)\frac{\partial^i u(x,t)}{\partial t^{i-j}\partial x^j}\,d\tau = f(x,t). \quad (1)$$

It is assumed that in the expansion

$$\sum_{j=0}^{m} a_{mj}(x,t)\lambda^{m-j}\xi^j = \prod_{j=1}^{m}(\lambda - \lambda_j(x,t)\xi)$$

the functions $\lambda_j(x,t)$ $(1 \le j \le m)$ are real and distinct for all $(x,t) \in \Pi$. We shall assume

$$\lambda_1(x,t) < \lambda_2(x,t) < \cdots < \lambda_m(x,t).$$

Let p of the functions $\lambda_j(x,t)$ be negative and q positive $(0 \le p \le m,\ 0 \le q \le m)$. $(p+q = m-1$, if one of the functions $\lambda_j(x,t)$ is equal to zero, and $p+q=m$, if all $\lambda_j(x,t)$ are different from zero.)

The problem is posed of finding, inside Π, a solution of equation (1) satisfying the initial conditions

$$\left.\frac{\partial^i u}{\partial t^i}\right|_{t=0} = g_i(x) \qquad (0 \le x \le l;\ 0 \le i \le m-1) \tag{2}$$

and the boundary conditions

$$\sum_{i=0}^{m-1}\sum_{j=0}^{i} \alpha_{ij}^s(t)\frac{\partial^i u(0,t)}{\partial t^{i-j}\partial x^j} + \int_0^t \sum_{i=0}^{m-1}\sum_{j=0}^{i} \alpha_{ij}^s(t,\tau)\frac{\partial^i u(0,\tau)}{\partial \tau^{i-j}\partial x^j}\,d\tau = h_s^1(t) \qquad (1 \le s \le p),$$

$$\sum_{i=0}^{m-1}\sum_{j=0}^{i} \beta_{ij}^s(t)\frac{\partial^i u(l,t)}{\partial t^{i-j}\partial x^j} + \int_0^t \sum_{i=0}^{m-1}\sum_{j=0}^{i} \beta_{ij}^s(t,\tau)\frac{\partial^i u(l,\tau)}{\partial \tau^{i-j}\partial x^j}\,d\tau = h_s^2(t) \qquad (1 \le s \le q). \tag{3}$$

It is assumed that the natural compatibility conditions of the initial and boundary conditions at the corner points $(0,0)$ and $(l,0)$ are fulfilled.

Theorem. Suppose the following conditions are satisfied:
a) the coefficients $a_{mj}(x,t)$ $(0 \le j \le m)$ are continuously differentiable in Π $m+1$ times (and, for all $(x,t) \in \Pi$, $a_{m0}(x,t) \ne 0$);
b) the coefficients $a_{ij}(x,t)$, $b_{ij}(x,t,\tau)$ $(0 \le i \le m-1,\ 0 \le j \le i)$ and the free term $f(x,t)$ are continuously differentiable in Π m times;
c) the initial functions $g_i(x)$ $(0 \le i \le m-1)$ are continuously differentiable, respectively, $2m-i-1$ times for $0 \le x \le l$;
d) the coefficients $\alpha_{ij}^s(t)$, $\alpha_{ij}^s(t,\tau)$, $\beta_{ij}^r(t)$, $\beta_{ij}^r(t,\tau)$ and the free terms $h_s^1(t)$ and $h_r^2(t)$ of the boundary conditions are continuously differentiable m times $(0 \le i \le m-1,\ 0 \le j \le i,\ 1 \le s \le p,\ 1 \le r \le q,\ 0 \le t \le T,\ 0 \le \tau \le t)$;
e)

$$D_1(t) = \begin{vmatrix} \sum\limits_{j=0}^{m-1} \alpha_{m-1,j}^1(t)\lambda_1^{m-j-1}(0,t) & \cdots & \sum\limits_{j=0}^{m-1} \alpha_{m-1,j}^1(t)\lambda_p^{m-j-1}(0,t) \\ \cdots & \cdots & \cdots \\ \sum\limits_{j=0}^{m-1} \alpha_{m-1,j}^p(t)\lambda_1^{m-j-1}(0,t) & \cdots & \sum\limits_{j=0}^{m-1} \alpha_{m-1,j}^p(t)\lambda_p^{m-j-1}(0,t) \end{vmatrix} \ne 0$$

$$(0 \le t \le T), \tag{4}$$

$$D_2(t) = \begin{vmatrix} \sum\limits_{j=0}^{m-1} \beta^1_{m-1,j}(t)\lambda^{m-j-1}_{m-q+1}(l,t) & \cdots & \sum\limits_{j=0}^{m-1} \beta^1_{m-1,j}(t)\lambda^{m-j-1}_{m}(l,t) \\ \cdots & \cdots & \cdots \\ \sum\limits_{j=0}^{m-1} \beta^q_{m-1,j}(t)\lambda^{m-j-1}_{m-q+1}(l,t) & \cdots & \sum\limits_{j=0}^{m-1} \beta^q_{m-1,j}(t)\lambda^{m-j-1}_{m}(l,t) \end{vmatrix} \neq 0$$

$$(0 \leq t \leq T).$$

Then in Π there exists a unique m-times continuously differentiable solution of equation (1), satisfying the initial conditions (2) and the boundary conditions (3).

The proof of the theorem is carried out as follows. By introducing the new unknown functions

$$v_{ij} = \frac{\partial^i u}{\partial t^{i-j} \partial x^j} \qquad (0 \leq i \leq m-1;\ 0 \leq j \leq i)$$

we reduce problem (1)–(3) to the equivalent problem of finding a solution of the first-order system (2):

$$\begin{gathered} \frac{\partial v_{m-1,0}}{\partial t} + \sum_{j=0}^{m-1} a_{m,j+1} \frac{\partial v_{m-1,j}}{\partial x} + \sum_{i=0}^{m-1} \sum_{j=0}^{i} a_{ij} v_{ij} = f, \\ \frac{\partial v_{m-1,r}}{\partial t} - \frac{\partial v_{m-1,r-1}}{\partial x} = 0 \qquad (1 \leq r \leq m-1), \\ \frac{\partial v_{ij}}{\partial t} - v_{i+1,j} = 0 \qquad (0 \leq i \leq m-2;\ 0 \leq j \leq i), \end{gathered} \tag{5}$$

satisfying the initial conditions

$$v_{ij}(x,0) = \frac{d^j g_{i-j}(x)}{dx^j} \qquad (0 \leq i \leq m-1;\ 0 \leq j \leq i) \tag{6}$$

and the boundary conditions

$$\sum_{i=0}^{m-1} \sum_{j=0}^{i} \alpha^s_{ij}(t) v_{ij}(0,t) + \int_0^t \sum_{i=0}^{m-1} \sum_{j=0}^{i} \alpha^s_{ij}(t,\tau) v_{ij}(0,\tau)\, d\tau = h^1_s(t) \qquad (1 \leq s \leq p)$$

(7)

$$\sum_{i=0}^{m-1}\sum_{j=0}^{i}\beta_{ij}^{s}(t)v_{ij}(l,t)+\int_0^t\sum_{i=0}^{m-1}\sum_{j=0}^{i}\beta_{ij}^{s}(t,\tau)v_{ij}(l,\tau)\,d\tau=h_s^2(t)\qquad(1\le s\le q)$$

Reducing system (5) to canonical form and introducing new unknown functions z_{ij} by the substitution

$$v_{m-1,j}=\sum_{k=1}^{m}\frac{\lambda_k^{m-j-1}}{\prod_{s\neq k}(\lambda_k-\lambda_s)}z_{mk}\qquad(0\le j\le m-1),$$

$$v_{ij}=z_{i+1,j+1}\qquad(0\le i\le m-2;\ 0\le j\le i),$$

we reduce problem (5)–(7) to the equivalent problem

$$\frac{\partial z_{ij}}{\partial t}=\lambda_{ij}\frac{\partial z_{ij}}{\partial x}+\sum_{r=1}^{m}\sum_{s=1}^{r}a_{ij}^{rs}z_{rs}+f_{ij}\qquad(1\le i\le m;\ 1\le j\le i);\tag{8}$$

$$z_{ij}(x,0)=\begin{cases}\displaystyle\sum_{r=0}^{m-1}\sum_{s=0}^{r}a_{ms}(x,0)\lambda_j^{r-s}(x,0)\frac{d^r g_{m-j-1}(x)}{dx^r}, & i=m;\ 1\le j\le i,\\[2ex] \displaystyle\frac{d^{j-1}g_{i-j}(x)}{dx^{j-1}}, & 1\le i\le m-1;\ 1\le j\le i;\end{cases}\tag{9}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{i}\gamma_{ij}^{s}(t)z_{ij}(0,t)+\int_0^t\sum_{i=1}^{m}\sum_{j=1}^{i}\gamma_{ij}^{s}(t,\tau)z_{ij}(0,\tau)\,d\tau=h_s^1(t)\qquad(1\le s\le p),$$

$$\sum_{i=1}^{m}\sum_{j=1}^{i}\delta_{ij}^{s}(t)z_{ij}(l,t)+\int_0^t\sum_{i=1}^{m}\sum_{j=1}^{i}\delta_{ij}^{s}(t,\tau)z_{ij}(l,\tau)\,d\tau=h_s^2(t)\qquad(1\le s\le q),\tag{10}$$

where

$$\lambda_{ij}=\begin{cases}\lambda_j, & i=m;\ 1\le j\le m,\\ 0, & 1\le i\le m-1;\ 1\le j\le i,\end{cases}$$

and the coefficients a_{ij}^{rs}, γ_{ij}^{s}, δ_{ij}^{s} and the free terms f_{ij} are expressed in an obvious way through the coefficients and the free term of equation (1) and the coefficients of the boundary conditions (3).

Problem (8)–(10), by integration along the characteristics (1), in view of conditions (4), is reduced to a system of Volterra integral equations solvable by the method of successive approximations.

2. In the special case when the boundary conditions have the form

$$\frac{\partial^i u(0,t)}{\partial x^i} = h_i^1(t) \qquad (0 \le i \le p-1); \qquad \frac{\partial^j u(l,t)}{\partial x^j} = h_j^2(t) \qquad (0 \le j \le q-1),$$

conditions (4) are always satisfied. Indeed, differentiating these conditions with respect to t, respectively $m-i-1$ and $m-j-1$ times $(0 \le i \le p-1;\ 0 \le j \le q-1)$ and comparing with (3), we obtain

$$\alpha_{ij}^{s}(t) = \delta_{m-1}^{i}\delta_{s}^{j+1}, \qquad \beta_{ij}^{r}(t) = \delta_{m-1}^{i}\delta_{r}^{j+1}, \qquad \alpha_{ij}^{s}(t,\tau) = \beta_{ij}^{r}(t,\tau) = 0$$

$$(0 \le i \le m-1;\ 0 \le j \le i;\ 1 \le s \le p;\ 1 \le r \le q)$$

(δ_i^j is the Kronecker symbol).

Then

$$D_1(t) = (-1)\, p(p+1)/2 \prod_{i=1}^{p} \lambda_i^{m-p}(0,t) \prod_{p \ge i > j \ge 1} [\lambda_i(0,t) - \lambda_j(0,t)] \ne 0,$$

$$D_2(t) = (-1)\, q(q+1)/2 \prod_{i=1}^{q} \lambda_{m-q+1}^{m-q}(l,t) \prod_{q \ge i > j \ge 1} [\lambda_{m-q+i}(l,t) - \lambda_{m-q+j}(l,t)] \ne 0.$$

The conditions (4) are essential for the unique solvability of the problem. One can give an example of a problem in which at least one of the conditions (4) is not satisfied and which, therefore, has an infinite number of solutions.

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named after Ivan Franko

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REFERENCES

1. V. E. Abolinya, A. D. Myshkis, *Uch. zap. Latv. univ.*, **20**, no. 3, 87 (1958).
2. I. G. Petrovskii, *Lectures on Partial Differential Equations*, Moscow, 1953.
3. L. Gårding, *UMN*, vol. 1 (91), 137 (1960).
4. L. L. Campbell, A. Robinson, *Proc. Lond. Math. Soc.*, (3), **5**, 129 (1955).
5. V. Thomée, *Math. Scand.*, **3**, no. 1, 115 (1955).
6. V. Thomée, *Math. Scand.*, **5**, no. 1, 93 (1957).
7. G. Peuser, *J. Math. and Mech.*, **6**, no. 5, 641 (1957).
8. Z. O. Melnyk, *Dokl. Akad. Nauk SSSR*, no. 9, 1149 (1963).

Note: Figure translations are in progress. See original paper for figures.

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