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Abstract

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THEORY OF ELASTICITY

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ON EXTREMAL PATHS OF PLASTIC DEFORMATION

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A plastically hardening material is considered. The initial and subsequent yield surfaces are determined by a loading function, which, generally speaking, is a complicated relation between the parameters of the state (^1). In particular, let us consider a loading function of the form

$$f(\sigma_{ij}, \varepsilon_{ij}, \chi) = 0, \tag{1}$$

where σ_{ij}, ε_{ij} are the tensors of stresses and plastic strains, and χ is a hardening parameter.

Among the various expressions for the hardening parameter, the length of the path in the space of plastic strains is often used,

$$\chi = \int_0^\tau \sqrt{\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}}\, dt. \tag{2}$$

The values of the loading parameter $t = 0$ and $t = \tau$ correspond to the initial and current instants of plastic deformation.

Assuming the yield surface to be smooth, we shall define the relation between the stresses and the rates of plastic strain by the associated flow law

$$\dot{\varepsilon}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \quad (\lambda > 0). \tag{3}$$

In the space of plastic strains E, consider all possible deformation paths, not containing unloading processes, which connect two points corresponding to two deformed states ε_{ij}^1 and ε_{ij}^2. The deformed state ε_{ij}^1 has been attained as a result of a certain loading path and corresponds to the loading parameter t_1. For the subsequent deformation paths connecting the points ε_{ij}^1 and ε_{ij}^2, the loading parameter can always be chosen so that, at $t = t_2$, $\varepsilon_{ij}(t_2) = \varepsilon_{ij}^2$. The dissipation of mechanical work on any of the indicated deformation paths is equal to

$$D = \int_{t_1}^{t_2} \sigma_{ij}\dot{\varepsilon}_{ij}\, dt.$$

Let us determine the path on which the dissipation attains its minimum value. The necessary condition for a minimum (^2) requires that

$$\delta D = \int_{t_1}^{t_2} (\sigma_{ij}\delta\dot{\varepsilon}_{ij} + \dot{\varepsilon}_{ij}\delta\sigma_{ij})\, dt. \tag{4}$$

Using the variation of relation (1),

$$\frac{\partial f}{\partial \sigma_{ij}}\delta\sigma_{ij} + \frac{\partial f}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij} + \frac{\partial f}{\partial \chi}\delta\chi = 0$$

and the associated flow law (3), condition (4) can be written in the form

$$\int_{t_1}^{t_2} \left(\sigma_{ij}\,\delta\dot{\varepsilon}_{ij} - \lambda\frac{\partial f}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij} - \lambda\frac{\partial f}{\partial \chi}\delta\chi\right) dt = 0. \tag{5}$$

Since the loading path on the interval $0 \leqslant t \leqslant t_1$ is assumed to be prescribed, the variation of relation (2) will have the form

$$\delta\chi = \int_{t_1}^{\tau} \frac{\dot{\varepsilon}_{ij}}{\sqrt{\dot{\varepsilon}_{kl}\dot{\varepsilon}_{kl}}}\,\delta\dot{\varepsilon}_{ij}\, dt.$$

In relation (5) we integrate by parts the term containing $\delta\chi$, and, since $\delta\varepsilon_{ij}(t_1) = \delta\varepsilon_{ij}(t_2) = 0$, we obtain

$$\int_{t_1}^{t_2} \left[\sigma_{ij}\,\delta\dot{\varepsilon}_{ij} - \lambda\frac{\partial f}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij} - \left(\int_{t}^{t_2} \lambda\frac{\partial f}{\partial \chi}\, dt\right)\frac{\dot{\varepsilon}_{ij}}{\sqrt{\dot{\varepsilon}_{kl}\dot{\varepsilon}_{kl}}}\,\delta\dot{\varepsilon}_{ij}\right] dt = 0. \tag{6}$$

After integration by parts of the terms containing $\delta\dot{\varepsilon}_{ij}$, under the integral sign in (6) there will be a common factor $\delta\varepsilon_{ij}$. For arbitrary variations $\delta\varepsilon_{ij}$, condition (6) will be satisfied if

$$\frac{d}{dt}\left[\sigma_{ij} - \left(\int_{t}^{t_2} \lambda\frac{\partial f}{\partial \chi}\, dt\right)\frac{\dot{\varepsilon}_{ij}}{\sqrt{\dot{\varepsilon}_{kl}\dot{\varepsilon}_{kl}}}\right] = -\lambda\frac{\partial f}{\partial \varepsilon_{ij}}. \tag{7}$$

The system of integro-differential equations (1), (3), (7) determines the loading path from the plastically deformed state ${}^1\varepsilon_{ij}$ to ${}^2\varepsilon_{ij}$ on which the minimum dissipation is attained. We note that if the system of equations (1), (3), (7) has a unique solution, it is not necessary to consider the sufficient condition for a minimum, since the maximum of dissipation is not bounded.

Let us find the condition which the loading function (1) must satisfy so that, on rectilinear trajectories in the space E, the dissipation of mechanical work assumes a minimum value. In this case the system of equations (1), (3), (7) must be equivalent to the condition of rectilinearity of the trajectories of plastic strains

$$\frac{d}{dt}\frac{\dot{\varepsilon}_{ij}}{\sqrt{\dot{\varepsilon}_{kl}\dot{\varepsilon}_{kl}}} = 0. \tag{8}$$

Using the associated flow law (3), we write relation (8) in the form

$$\left(\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}}\right)\left(\frac{\partial^2 f}{\partial \sigma_{ij}\partial \sigma_{mn}}\dot{\sigma}_{mn} + \frac{\partial^2 f}{\partial \sigma_{ij}\partial \varepsilon_{mn}}\dot{\varepsilon}_{mn} + \frac{\partial^2 f}{\partial \sigma_{ij}\partial \chi}\dot{\chi}\right) =$$

$$= \frac{\partial f}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{kl}}\left(\frac{\partial^2 f}{\partial \sigma_{kl}\partial \sigma_{mn}}\dot{\sigma}_{mn} + \frac{\partial^2 f}{\partial \sigma_{kl}\partial \varepsilon_{mn}}\dot{\varepsilon}_{mn} + \frac{\partial^2 f}{\partial \sigma_{kl}\partial \chi}\dot{\chi}\right). \tag{9}$$

On rectilinear trajectories of plastic strains, the extremality condition (7) takes the form

$$\dot{\sigma}_{ij} = -\lambda\left(\frac{\partial f}{\partial \varepsilon_{ij}} + \frac{\partial f}{\partial \chi}\frac{\dot{\varepsilon}_{ij}}{\sqrt{\dot{\varepsilon}_{kl}\dot{\varepsilon}_{kl}}}\right). \tag{10}$$

Introduce the notation

$$L_{ij} = \frac{1}{\lambda}\left(\frac{\partial^2 f}{\partial \sigma_{ij}\partial \sigma_{mn}}\dot{\sigma}_{mn} + \frac{\partial^2 f}{\partial \sigma_{ij}\partial \varepsilon_{mn}}\dot{\varepsilon}_{mn} + \frac{\partial^2 f}{\partial \sigma_{ij}\partial \chi}\dot{\chi}\right). \tag{11}$$

Substituting into this expression the values of $\dot{\sigma}_{mn}$, $\dot{\varepsilon}_{mn}$, $\varkappa$ from relations (2), (3), (10), we obtain

$$L_{ij} = \frac{\partial^2 f}{\partial \sigma_{ij}\partial \sigma_{mn}}\left(-\frac{\partial f}{\partial \varkappa}\frac{\partial f}{\partial \sigma_{mn}}\right)\sqrt{\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}} - \frac{\partial f}{\partial e_{mn}}} + \frac{\partial^2 f}{\partial \sigma_{ij}\partial e_{mn}}\frac{\partial f}{\partial \sigma_{mn}} + \frac{\partial^2 f}{\partial \sigma_{ij}\partial \varkappa}\sqrt{\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}}}. \tag{12}$$

Relation (9) can be represented in the form

$$\left(\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}}\right)L_{ij} = \frac{\partial f}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{kl}}L_{kl}, \tag{13}$$

where L_{ij} must be substituted from relation (12). Thus, the trajectories of plastic strains corresponding to a minimum of dissipation will be rectilinear if the loading function (1) satisfies condition (13).

Let us consider the case of two-dimensional shear. The stresses σ_{13}, σ_{23} and plastic strains ε_{13}, ε_{23} will be denoted by σ_i, ε_i $(i = 1, 2)$. We assume the material to be isotropic, i.e., the loading function must depend on the invariants $\alpha = {}^1/_2\sigma_i\sigma_i$, $\beta = \sigma_i\varepsilon_i$, $\gamma = {}^1/_2\varepsilon_i\varepsilon_i$, $\varkappa = \int \sigma_i d\varepsilon_i$. In this case the loading surface can always be represented in the form

$$\alpha + F(\beta, \gamma, \varkappa) = 0. \tag{14}$$

Condition (13) for two-dimensional shear is written in the form

$$\left(\frac{\partial f}{\partial \sigma_j}\frac{\partial f}{\partial \sigma_j}\right)L_{i3} = \frac{\partial f}{\partial \sigma_i}\frac{\partial f}{\partial \sigma_j}L_{j3}. \tag{15}$$

This condition can be regarded as a vector equation, which will be satisfied if

$$L_{i3} = 0 \tag{16}$$

or

$$\frac{\partial f}{\partial \sigma_i} = \mu L_{i3}, \tag{17}$$

where the latter relation is the condition of collinearity of the vectors.

We shall represent condition (17), imposed on the loading surface (14), in the form of a system of differential equations with respect to the function F. Let us compute the necessary derivatives of F, which we shall denote by the corresponding subscript below,

$$\frac{\partial f}{\partial \sigma_i} = \sigma_i + F_\beta\varepsilon_i, \qquad \frac{\partial f}{\partial e_i} = F_\beta\sigma_i + F_\gamma\varepsilon_i,$$

$$\frac{\partial^2 f}{\partial \sigma_i \partial \sigma_j} = \delta_{ij} + F_{\beta\beta}\varepsilon_i\varepsilon_j, \qquad \frac{\partial^2 f}{\partial \sigma_i \partial e_j} = \delta_{ij}F_\beta + F_{\beta\beta}\varepsilon_i\sigma_j + F_{\beta\gamma}\varepsilon_i\varepsilon_j. \tag{18}$$

Substituting expressions (18) into relations (12), (17) and multiplying scalarly by the vectors σ_i, ε_i, we obtain a system of equations in the variables α, β, γ

$$\alpha+F_\beta\beta=\mu\left\{\frac{F_\varkappa}{\sqrt{\alpha+2F_\beta\beta+F_\beta^2\gamma}}\left[\alpha+F_\beta\beta+F_{\beta\beta}(\beta+F_\beta\gamma)\right]+\right.$$

$$\left.+(F_\beta^2-F_\gamma)\beta+F_{\beta\beta}(\alpha-F_\gamma\gamma)\beta+F_{\beta\gamma}(\beta+F_\beta\gamma)+F_{\beta\varkappa}\beta\sqrt{\alpha+F_\beta\beta+F_\beta^2\gamma}\right\},$$

$$\beta+F_\beta\gamma=\mu\left\{-\frac{F_\varkappa}{\sqrt{\alpha+2F_\beta\beta+F_\beta^2\gamma}}\left[\beta+F_\beta\gamma+F_{\beta\beta}(\beta+F_\beta\gamma)\gamma\right]+\right.$$

$$\left.+(F_\beta^2-F_\gamma)\gamma+F_{\beta\beta}(\alpha-F_\gamma\gamma)\gamma+F_{\beta\gamma}(\beta+F_\beta\gamma)\gamma+F_{\beta\varkappa}\gamma\sqrt{\alpha+2F_\beta\beta+F_\beta^2\gamma}\right\}. \tag{19}$$

Since the function F does not depend on α, in the equations obtained after the elimination of μ one must equate to zero the coefficients standing before the linearly independent functions of α. As a result we obtain the system of equations

$$F_{\beta\beta}=0,\qquad F_{\beta\chi}=0,\qquad F_\beta^2-F_\gamma+F_{\beta\gamma}\left(\beta+F_\beta\gamma\right)=0. \tag{20}$$

Solving the system of equations (20) leads to the following form of the loading surface:

$$\alpha+\Phi(\gamma)\beta+\Psi(\gamma,\chi)=0. \tag{21}$$

The functions Φ and Ψ must satisfy the equation

$$\Phi^2-\Psi_\gamma+\Phi\Phi_\gamma\gamma=0. \tag{22}$$

Using the independence of the function Φ from χ, we obtain the integral of equation (22)

$$\Psi=\Phi^2(\gamma)\gamma+R(\chi), \tag{23}$$

where the functions $\Phi(\gamma)$ and $R(\chi)$ may be chosen arbitrarily. Substituting the value of the function Ψ from relation (23), the loading surface (21) can be transformed to the form

$$(\sigma_i+\Phi\varepsilon_i)^2+2R(\chi)=0.$$

In the process of plastic flow the yield surface will remain a circle. The hardening law obtained represents nonlinear kinematic hardening ([3,4]) with simultaneous expansion of the yield surface.

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CITED LITERATURE

[1] L. I. Sedov, *Introduction to Continuum Mechanics*, 1962. [2] I. M. Gelfand, S. V. Fomin, *Calculus of Variations*, 1961. [3] A. Yu. Ishlinskii, *Ukr. Matem. Zhurn.*, 6, No. 3 (1954). [4] I. Yu. Kadashevich, V. V. Novozhilov, *DAN*, 117, No. 4 (1957).

Note: Figure translations are in progress. See original paper for figures.