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Abstract

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MATHEMATICS

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ON THE APPLICATION OF THE METHOD OF FRACTIONAL POWERS OF OPERATORS TO THE INVESTIGATION OF THE NAVIER–STOKES EQUATIONS

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1. The present paper* continues the investigation begun in ([2,3]). The system of Navier–Stokes equations is reduced to the integral equation

$$\mathbf{v}(t)=e^{-t\nu A}\mathbf{v}(0)+\int_0^t e^{-(t-s)\nu A}P\mathbf{f}\,ds-\int_0^t e^{-(t-s)\nu A}P\frac{\partial}{\partial x_k}(v_k\cdot\mathbf{v})\,ds, \qquad (1)$$

in the Hilbert space H–the closure in the metric $L_2(\Omega)$ of the set of smooth solenoidal m-dimensional vector functions in a bounded m-dimensional domain Ω, vanishing near the boundary S of the domain Ω. Here P is the operator of orthogonal projection in L_2 onto H; A is the Friedrichs extension of the operator $-P\Delta$, initially defined on $W_2^2\cap H$ ([1–3]).

Equation (1) has unbounded nonlinearities. Having made, in §§ 3, 4, and 5, a substitution, we shall pass to an integral equation with continuous nonlinearities and establish existence theorems and estimates for solutions of these equations. The estimates of A and of the nonlinearities needed for this are given in § 2 (cf. ([2,3])).

2. Lemma 1. For every $\alpha\in[0,1]$ the inequalities

$$\|(-\Delta)^\alpha\mathbf{v}\|\le C(\alpha)\|A^\alpha\mathbf{v}\| \quad (\mathbf{v}\in D[A^\alpha]); \qquad (2)$$

$$\|A^\alpha P\mathbf{v}\|\le C(\alpha)\|(-\Delta)^\alpha\mathbf{v}\| \quad (\mathbf{v}\in D[(-\Delta)^\alpha]). \qquad (3)$$

hold.

Proof. In ([2]), (2) was established for $\alpha=2^{-n}$, $n=1,2,...$, with $C(\alpha)=1$. It is not hard to see that $P\mathbf{u}=\mathbf{u}-\operatorname{grad}\Delta^{-1}\operatorname{div}\mathbf{u}-\operatorname{grad}q$, where $\Delta q=0$ and

$$\frac{\partial q}{\partial n}\bigg|_S = (\mathbf{u} - \frac{\partial}{\partial n}\Delta^{-1}\operatorname{div}\mathbf{u})\bigg|_S .$$

By virtue of the known properties of solutions of the Neumann problem (see ([4])),

$$\|q\|_{W_2^i} \le C_i\|\mathbf{u} - \operatorname{grad}\Delta^{-1}\operatorname{div}\mathbf{u}\|_{W_2^{i-1}}, \quad i = 1, 2, \dots .$$

Therefore $P\mathbf{u} \in W_2^i \cap H$, if $\mathbf{u} \in W_2^i$, and

$$\|P\mathbf{u}\|_{W_2^i} \le C_i\|\mathbf{u}\|_{W_2^i}.$$

By virtue of ([5]),

$$\|A\mathbf{u}\| \le C\|\mathbf{u}\|_{W_2^2} \quad (\mathbf{u} \in D[A]).$$

By virtue of ([6]),

$$\|\mathbf{u}\|_{W_2^2} \le C\|(-\Delta)\mathbf{u}\| \quad (\mathbf{u} \in D[(-\Delta)]).$$

Hence (3) holds for $\alpha = 1$. This means that

$$\|(-\Delta)^{-1}\mathbf{u}\| \le C(1)\|A^{-1}\mathbf{u}\| \quad (\mathbf{u} \in H).$$

Proceeding as in the proof of Lemma 2 of ([2]), we obtain

$$\|(-\Delta)^{-\alpha}\mathbf{u}\| \le [C(1)]^\alpha\|A^{-\alpha}\mathbf{u}\| \quad (\mathbf{u} \in H)$$

for $\alpha = 2^{-n}$, $n = 1, 2, \dots$. Thus, we have (3) for $\alpha = 2^{-n}$ with $C(\alpha) = [C(1)]^\alpha$. With the aid of this one can obtain (2) for all dyadic rational α. Let, for example, $\alpha = 3/4$. Since

$$I = (A^{-3/4}P[-\Delta]^{3/4}\vec{\varphi}, \vec{\psi}) = ([-\Delta]^{1/4}\vec{\varphi}, A^{-1}A^{1/4}\vec{\psi}),$$

$\vec{\varphi} \in D([-\Delta]^{3/4})$, $\vec{\psi} \in D(A^{1/4})$, then, taking into account that

$$\|A^{-1}P\mathbf{u}\| \le C\|[-\Delta]^{-1}\mathbf{u}\|$$

and applying Heinz’ s inequality ([7]) and (3) for $\alpha = 1/4$, we obtain

$$I \le C\|[-\Delta]^{-3/4}[-\Delta]^{3/4}\vec{\varphi}\| \cdot \|[-\Delta]^{-1/4}A^{1/4}\vec{\varphi}\| \le C\|\vec{\varphi}\|[C(1)]^{1/2}\|\vec{\psi}\|.$$

$C(\alpha)$ in (2)

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* These results were reported at the First Interzonal Scientific Conference in Voronezh in February 1962.

is thereby uniformly bounded. Passing to the limit we obtain from this (2) for any $\alpha \in [0, 1]$. Similarly, (3) is established.

By $G(m)$ denote the set of points of the plane (α, β) lying in the triangle with vertices $(0, 0)$, $((m+2)/8, (m+2)/8)$, and $(0, (m+2)/4)$, except for the points $(0, 0)$, $(0, (m+2)/4)$ and (α, β) with $\beta < 1/2$ and $\alpha > m/4 - 1/2$.

Lemma 2. For any $(\alpha, \beta) \in G(m)$, $\mathbf{v} \in D([1-\Delta]^{\alpha})$, $\mathbf{w} \in D([-\Delta]^{\beta})$, $\operatorname{div} \mathbf{v} = 0$, if $\alpha + \beta > m/4$,

$$\left\|(-\Delta)^{\alpha+\beta-(m+2)/4} \frac{\partial}{\partial x_k}(v_k \mathbf{w})\right\| \le C(\alpha, \beta)\|(-\Delta)^{\alpha}\mathbf{v}\| \cdot \|(-\Delta)^{\beta}\mathbf{w}\|. \qquad (4)$$

Proof. If $\alpha + \beta \le m/4$, then for any $\vec{\varphi} \in L_2$

$$I = \left|\int_{\Omega} \vec{\varphi}\,(-\Delta)^{\alpha+\beta-(m+2)/4} \frac{\partial}{\partial x_k}(v_k \mathbf{w})\,dx\right| \le \int_{\Omega} \left|\frac{\partial}{\partial x_k}(-\Delta)^{\alpha+\beta-(m+2)/4}\vec{\varphi}\right| \cdot |v_k \mathbf{w}|\,dx \le$$

$$\le m\left\|\operatorname{grad}(-\Delta)^{\alpha+\beta-(m+2)/4}\vec{\varphi}\right\|_{L_{\frac{m}{2(\alpha+\beta)}}} \cdot \|\mathbf{v}\|_{L_{\frac{2m}{m-4\alpha}}} \cdot \|\mathbf{w}\|_{L_{\frac{2m}{m-4\beta}}}.$$

Since

$$\|\mathbf{v}\|_{L_{\frac{2m}{m-4\gamma}}} \le C_{\gamma}\|(-\Delta)^{\gamma}\mathbf{v}\| \quad (0 < \gamma < m/4),$$

$$\|\operatorname{grad} \mathbf{v}\|_{L_{\frac{2m}{m+2-4\gamma}}} \le C_{\gamma}\|(-\Delta)^{\gamma}\mathbf{v}\| \quad (1/2 \le \gamma < (m+2)/4) \quad (\mathbf{v} \in D[(-\Delta)^{\gamma}])$$

(see $(2, 8)$), it follows from this that (4) holds. If $\alpha + \beta > m/4$, then $\beta > 1/2$, and

$$I \le \int_{\Omega} \left|(-\Delta)^{\alpha+\beta-(m+2)/4}\vec{\varphi}\right| \left|v_k \frac{\partial \mathbf{w}}{\partial x_k}\right| dx \le$$

$$\leq m \left\|(-\Delta)^{\alpha+\beta-(m+2)/4}\vec{\varphi}\right\|_{L_{\frac{m}{2(\alpha+\beta)-1}}} \cdot \|\mathbf{v}\|_{L_{\frac{2m}{m-4\alpha}}} \|\operatorname{grad}\mathbf{w}\|_{L_{\frac{2m}{m+2-4\beta}}}.$$

Hence, and from $(2, 8)$, (4) follows.

Remark. From Lemma 1 it follows that in (4) one may replace $(-\Delta)$ by A.

3. Let $m = 2, 3$. Making in (1) the substitution $\mathbf{w}(t, \mu, \gamma) = t^{\mu-\gamma}A^{\mu}\mathbf{v}(t)$ $(\mu = \alpha, \beta;\ (\alpha, \beta) \in G(m);\ (m-2)/4 \leq \gamma \leq \alpha;\ \alpha > (m-2)/4)$, we obtain

$$\begin{aligned}\mathbf{w}(t,\mu,\gamma) = t^{\mu-\gamma}A^{\mu}e^{-t\nu A}\mathbf{v}(0) + t^{\mu-\gamma}A^{\mu}\int_0^t e^{-(t-s)\nu A}P\mathbf{f}\,ds \\ - t^{\mu-\gamma}\int_0^t A^{\mu+(m+2)/4-(\alpha+\beta)}e^{-(t-s)\nu A}A^{\alpha+\beta-(m+2)/4}P\frac{\partial}{\partial x_k}\big[A^{-\alpha}w_k(s,\alpha,\gamma)\times \\ \times A^{-\beta}\mathbf{w}(s,\beta,\gamma)\big]s^{2\gamma-(\alpha+\beta)}\,ds.\end{aligned} \tag{5}$$

By $\vec{\varphi}_i$ denote the i-th term of the right-hand side of (5). Let

$$\mathbf{v}(0) \in D(A^{\gamma}). \tag{*}$$

Then $\vec{\varphi}_1$ is continuous in t for $t \geq 0$. If $\gamma < \mu$, then $\vec{\varphi}_1 = 0$ at $t = 0$. Let

$$\vec{\varphi}_2 \text{ be continuous in } t \in [0,\tau] \text{ for some } \tau > 0 \text{ and } \vec{\varphi}_2 = 0 \text{ at } t = 0. \tag{**}$$

If

$$\mathbf{w}(t,\mu,\gamma) \text{ are continuous on } [0,\tau]\ (\mu = \alpha, \beta), \tag{***}$$

then $\vec{\varphi}_3$ is also continuous on $[0, \tau]$, and $\vec{\varphi}_3 = 0$ at $t = 0$*.

* To prove the latter for $\gamma = (m-2)/4$, it suffices to consider only differentiable $\mathbf{w}(t, \mu, \gamma)$.

All this makes it possible to consider system (5) as an operator equation in the space $C[0, \tau]$—the space of vector functions $[\mathbf{w}^1(t), \mathbf{w}^2(t)]$ continuous on $[0, \tau]$, with values in $L_2 \times L_2$ and norm

$$\|[\mathbf{w}^1(t), \mathbf{w}^2(t)]\| = \max_{0,\tau;\, i}\|\mathbf{w}^i(t)\|.$$

Let $S[R]$ be the ball of the space $C[0, \tau]$ with center at zero and radius R. The right-hand side of system (5) defines a nonlinear operator $\mathcal{L}$, acting and continuous in $C[0, \tau]$ (see (4)). If $MN < 1/4$, where

$$M = M(\tau, \alpha, \beta, \gamma) = \max\|\vec{\varphi}_1 + \vec{\varphi}_2\|,$$

$$N = N(\tau, \alpha, \beta, \gamma) = \max t^{\mu-\gamma} \int_0^t \|A^{\mu+(m+2)/4-(\alpha+\beta)} e^{-(t-s)\nu A}\| s^{2\gamma-(\alpha+\beta)}\, ds \cdot C(\alpha, \beta) \tag{6}$$

(the maximum is taken over $t \in [0, \tau]$ and $\mu = \alpha, \beta$), then in the ball $S[(1 - \sqrt{1-4MN})(2N)^{-1}]$ the operator $\mathcal{L}$ is a contraction. Consequently, in this ball there exists a unique solution of system (5), which can be found by the method of successive approximations. The formula $\mathbf{v}(t) = t^{\gamma-\mu} A^{-\mu} \mathbf{w}(t, \mu, \gamma)$ determines a solution of (1) and, consequently, a solution of the system of Navier–Stokes equations (see ([2,3])). The solution (5) found is unique in $C[0, \tau]$. For the proof in the case $\gamma > (m-2)/4$, see ([2,3]). If $\gamma \in [(m-2)/4, \mu)$, then, by virtue of *(), () **and** ()*, $\mathbf{w}(0, \mu, \gamma) = 0$. Since the nonlinearity in (5) is of second order, uniqueness holds on a small interval. Next one must pass to equation (1) and apply the "gluing lemma" (see, for example, ([9])). Thus one proves:

Theorem 1. *System (5) has in $C[0, \tau]$ a unique solution if ()* , (**) and*

$$MN < 1/4. \tag{7}$$

By virtue of *(), () **and** ()* condition (7) is satisfied for small $\tau > 0$—the local existence theorem. We note that it is established for $\mathbf{v}(0) \in D(A^{(m-2)/4})$. If (7) is satisfied for large $\tau > 0$ or for $\tau = \infty$, then we obtain a nonlocal theorem more general than the existing ones (see ([10,11]).

Theorem 2. *Under the conditions of Theorem 1*

$$\|\mathbf{w}(t, \mu, \gamma)\| \leq M \qquad (0 \leq t \leq \tau;\ \mu = \alpha, \beta). \tag{8}$$

This makes it possible to study the stability of the trivial solution with respect to perturbations of the initial velocities $\mathbf{v}(0)$ and forces $\mathbf{f}$. By Lemma 2, this stability can be estimated in the norms of the Aronszajn–Slobodetskii spaces (see ([4]).

4. A nonlocal theorem for (5) can be obtained with the aid of the local one if one establishes the existence of such an R that from the inequality $\|\mathbf{w}\| \leq R$ on $[0, \tau]$ there follows the inequality $\|\mathbf{w}\| \leq R_1 < R$. Let

$$\vec{\varphi}_2 \text{ for } \mu = 1/2 \text{ be continuous for } t > 0. \tag{****}$$

Then (cf. ([2,3])) for solutions of (1) the energy inequality is valid, whence

$$\|\mathbf{v}(t)\| \leq e^{-\delta t} \|\mathbf{v}(0)\| + \int_0^t e^{-\delta(t-s)} \|P\mathbf{f}\|\, ds \leq V = V(\tau)$$

$$(0 \leq t \leq \tau,\ \delta = \inf \sigma(-\Delta),\ \sigma(-\Delta) \text{ is the spectrum of } (-\Delta)). \tag{9}$$

From (5), the moment inequality (12), and (9) it follows that

$$W(\tau, \beta, \gamma) \leq M(\tau, \beta, \beta, \gamma) + K(\tau, \alpha, \beta, \gamma) V(\tau)^{1-\alpha/\beta} W(M, \beta, \gamma)^{1+\alpha/\beta}, \tag{10}$$

where

$$W = W(\tau, \beta, \gamma) = \max_{[0,\tau]} \|w(t, \beta, \gamma)\|,$$

$$K = K(\tau, \alpha, \beta, \gamma) = \max_{[0,\tau]} t^{\beta-\gamma} \int_0^t \|A^{(t+2)/4-\alpha} e^{-(t-s)\nu A}\| \, s^{(\gamma-\beta)(1+\alpha/\beta)} C(\alpha, \beta) \, ds.$$

From (10) we obtain

Theorem 3. Suppose that $(****)$ is satisfied and that

$$KV^{1-\alpha/\beta} M^{\alpha/\beta} < \frac{\beta}{\beta+\alpha} \left(\frac{\alpha}{\beta+\alpha}\right)^{\alpha/\beta}. \tag{11}$$

Then (5) has a unique solution in $C[0, \tau]$, and

$$\|w(t, \beta, \gamma)\| \leq Mr, \tag{12}$$

where r is the smallest root of the equation

$$r = 1 + KV^{1-\alpha/\beta} M^{\alpha/\beta} r^{1+\alpha/\beta}.$$

5. **Theorem 4.** Suppose that $(****)$ is satisfied and $m = 2$. Then

$$\|w(t, \alpha, 0)\| \leq Q(\tau, \alpha) \qquad (0 < \alpha < 1/2), \tag{13}$$

where $Q(\tau, \alpha)$ depends only on

$$\|v(0)\|, \qquad \int_0^t \|Pf\| \, ds, \qquad \max_{t \in [0,\tau]} t^\alpha \left\| A^\alpha \int_0^t e^{-(t-s)\nu A} Pf \, ds \right\|.$$

Proof. Using the estimate (9)

$$\left\| A^{1/2} \int_0^t e^{-(t-s)A} f(s) \, ds \right\| \leq \frac{1}{\sqrt{2}} \left[\int_0^t \|f(s)\|^2 \, ds \right]^{1/2},$$

which is valid for any positive definite self-adjoint operator, the system (5) for $\beta = 1/2$, and estimate (4), we obtain

$$\|w(t,\alpha,0)\| \le \|t^\alpha A^\alpha e^{-t\nu A} v(0)\| + \left\| t^\alpha A^\alpha \int_0^t e^{-(t-s)\nu A} Pf\, ds \right\|$$
$$+ t^\alpha \int_0^{t/2} \|A^{1/2} e^{-(t-s)\nu A}\| C(\alpha,1/2) \|w(s,\alpha,0)\|\, \|A^{1/2} v(s)\|\, s^{-\alpha}\, ds$$
$$+ \frac{2^{\alpha-1/2} C(\alpha,1/2)}{\sqrt{\nu}} \left[\int_{t/2}^t \|w(s,\alpha,0)\|^2 \|A^{1/2} v(s)\|^2 s^{-2\alpha}\, ds \right]^{1/2},$$

where $v(s)$ is the solution of (1). Since, by virtue of the energy inequality,

$$\int_0^t \|A^{1/2} v\|\, ds$$

is finite, (13) follows from this.

Estimate (13) means that for the system (5) a local existence theorem holds for any $v(0) \in H$.

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Note: Figure translations are in progress. See original paper for figures.