MATHEMATICS

A. Yu. LEVIN

1964

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. Yu. LEVIN

THE FREDHOLM EQUATION WITH A SMOOTH KERNEL AND BOUNDARY-VALUE PROBLEMS FOR A LINEAR DIFFERENTIAL EQUATION

(Presented by Academician I. G. Petrovskii, 15 V 1964)

1. We shall first formulate a result concerning the Fredholm equation

$$x(t) = \lambda \int_a^b K(t,s)x(s)\,d\mu(s) \qquad (a \leqslant t \leqslant b). \tag{1}$$

Here $\mu(t)$ is a function of bounded variation: $V_a^b \mu(t) \leqslant C_1$ (up to item 6 all functions are assumed to be complex-valued), and $K(t,s)$ is measurable and, for almost every s in $[a,b]$, satisfies the conditions

$$\|K(t,s)\| \leqslant C_2, \qquad V_a^b K^{(m)}(t,s) \leqslant C_3. \tag{2}$$

($m > 0$; differentiation and variation here and below are with respect to the first argument.) The second of inequalities (2) is understood in the sense that $K^{(m)}(t,s)$ almost everywhere on the interval $a \leqslant t \leqslant b$ coincides with a function whose variation on this interval is $\leqslant C_3$. Continuity of $K^{(m)}(t,s)$ is not required; it is assumed only that $K^{(m-1)}(t,s)$ is absolutely continuous in t for almost all s.

Theorem 1. *Under the stated assumptions, the numerator $D(t,s,\lambda)$ (for almost all s) and the denominator $D(\lambda)$ of the Fredholm resolvent for (1) are entire functions of λ of order not exceeding $\frac{1}{m+1}$, so that, in particular,*

$$D(\lambda) = \prod_i \left(1 - \frac{\lambda}{\lambda_i}\right), \tag{3}$$

where λ_i ($0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$) are the eigenvalues (e.v.) of equation (1), numbered with multiplicities taken into account. Moreover, all e.v. satisfy the inequality

$$|\lambda_k| \geqslant C k^{m+1}, \qquad C = C(C_1, C_2, C_3, m, b-a) > 0. \tag{4}$$

Of course, (4) is meaningful only in the case where e.v. exist. An explicit expression for C in terms of $C_1, C_2, C_3, m, b-a$ can be indicated. In the proof of Theorem 1, certain devices set forth in ([2]) are used, in combination with known results on the rate of decrease of the Fourier coefficients of smooth functions.

2. Let us consider a general nonsingular boundary-value problem for an equation of order n:

$$Lx \equiv x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = f(t) \qquad (a \leqslant t \leqslant b), \tag{5}$$

$$l_i[x] \equiv \sum_{k=0}^{n-2} \alpha_{ik} x^{(k)}(a) + \int_a^b x^{(n-1)}(t)\, dg_i(t) = \beta_i, \qquad i = 1, 2, \dots, n. \tag{6}$$

Here $p_1(t), \dots, p_n(t), f(t)$ are summable on $[a, b]$; $g_1(t), \dots, g_n(t)$ have bounded variation and are continuous from the right inside (a, b); α_{ik} and β_i are complex constants. The functionals $l_i[x]$ are written in a "reduced" form, which is convenient and at the same time general, since $x^{(k)}(t)$, $k = 0, 1, \dots, n-2$, are expressed linearly in terms of $x^{(k)}(a)$ and $x^{(n-1)}(t)$. By linearly combining the l_i, one can also simplify the matrix $\|\alpha_{ik}\|$.

If the corresponding homogeneous problem has no nontrivial solutions, then, as is known, there exists a Green' s function $G(t, s)$, defined in the square $K : a \leqslant t, s \leqslant b$, which makes it possible to write the solution of (5)–(6) in the form

$$x(t) = \int_a^b G(t, s) f(s)\, ds + \sum_{i=1}^{n} \beta_i y_i(t),$$

where $y_1(t), \dots, y_n(t)$ is a fundamental system of solutions of $Ly = 0$ such that $l_i[y_j] = \delta_{ij}$ $(i, j = 1, 2, \dots, n)$. Let $Y(t, s)$, as a function of t, be a solution of $Ly = 0$ such that, for $t = s$,

$$Y^{(i)}(t, s) = \delta_{i,n-1}, \quad i = 0, 1, \dots, n-1.$$

Further, let $H(t, s)$ be the Cauchy function for L, i.e. $H(t, s) = 0$ for $t \leqslant s$, and $H(t, s) = Y(t, s)$ for $t > s$. The formula

$$G(t, s) = H(t, s) - \sum_{i=1}^{n} y_i(t)\, l_i[H(t, s)]$$
$$= H(t, s) - \sum_{i=1}^{n} y_i(t) \int_s^b Y^{(n-1)}(t, s)\, dg_i(t) \tag{7}$$

is convenient.

It makes it possible, in particular, to investigate the properties of $G(t,s)$ as a function of s. It is easy to see, for example, that

$$G(t,s_0+0)-G(t,s_0-0)=\sum_{i=1}^{n} y_i(t)\,[g_i(s_0+0)-g_i(s_0-0)]$$

$$(a<s_0<b)$$

(for $n=1,\ t=s_0$, 1 must be subtracted from the right-hand side). Consequently, $G(t,s)$ is continuous in s on the line $s=s_0$ $(a<s_0<b,\ n>1)$ if and only if all $g_i(t)$ are continuous at the point $t=s_0$. Since $G(t,a)$ and $G(t,b)$ can be defined by continuity, the problem (5)–(6) for $n>1$ has a Green' s function continuous in K if and only if $g_1(t),\dots,g_n(t)$ are continuous inside (a,b). Under certain requirements on the smoothness of $p_i(t)$, one can also characterize the smoothness of $G(t,s)$ with respect to s, which, as it turns out, is also determined by the smoothness of $g_i(t)$. For what follows, however, what is essential is the smoothness of $G(t,s)$ with respect to t, independent of the boundary conditions, characterized by the fact that the variation of $G^{(n-1)}(t,s)$ on the interval $a\leqslant t\leqslant b$ is bounded uniformly in s.

Lemma 1. For all $s,\ a\leqslant s\leqslant b$, the inequality

$$\int_a^b |Y^{(n)}(t,s)|\,dt\leqslant\gamma-1,\quad \text{where}\ \ \gamma=\exp\left\{\sum_{k=1}^{n}\frac{(b-a)^{k-1}}{(k-1)!}\int_a^b |p_k(t)|\,dt\right\}.$$

holds.

It follows from this that $|Y^{(n-1)}(t,s)|\leqslant\gamma$, which, together with (7), gives an estimate, uniform in s, for $V_a^b G^{(n-1)}(t,s)$. At the same time we also obtain the estimate

$$|Y(t,s)|\leqslant\frac{\gamma}{(n-1)!}|t-s|^{n-1}.$$

3. Let us now consider the problem

$$Lx=\lambda q(t)x,\qquad l_i[x]=0,\quad i=1,2,\dots,n, \tag{8}$$

where $q(t)$ is summable on $[a,b]$. This problem is equivalent to the equation

$$x(t)=\lambda\int_a^b G(t,s)q(s)x(s)\,ds, \tag{9}$$

which, by what has been said, for $n > 1$ satisfies the conditions of Theorem 1 with $m = n - 1,\ d\mu(t) = q(t)\,dt$. Therefore $D(t, s, \lambda)$ and $D(\lambda)$ are functions of order no greater than $1/n$ (this is also true for $n = 1$); for $n > 1$ the expansion (3) holds, and all eigenvalues λ_k satisfy the inequality $|\lambda_k| \geqslant Ck^n$ $(C > 0)$. If the fundamental system for $Ly = 0$ is known, then C can be indi-

but effectively. We note that using Theorem 1 from (2) would lead here to substantially less accurate results.

Equating in the expansion (3) for (9) the coefficients of λ, we obtain the following formula (for $n > 1$):

$$\sum_i \frac{1}{\lambda_i} = \int_a^b G(t,t)q(t)\,dt. \tag{10}$$

4. What follows is connected with Sturm—Liouville problems of the following form ($q(t) \neq 0$ summable):

$$x^{(n)} + \lambda q(t)x = 0 \qquad (a \le t \le b), \tag{11}$$

$$x(a) = x'(a) = \cdots = x^{(n-k-1)}(a) = x(b) = x'(b) = \cdots = x^{(k-1)}(b) = 0 \qquad (1 \le k \le n-1). \tag{12}$$

Below $G(t, s)$ denotes the Green function of the operator $x^{(n)}$ under the conditions (12). Determining the form of $G(t, t)$ (which is not a trivial task in view of the arbitrariness of k, n) and using (10), we arrive at the proposition:

Theorem 2. The eigenvalues of the problem (11)—(12) satisfy the relation

$$\sum_i \frac{1}{\lambda_i} = \frac{(-1)^{k+1}}{(n-1)(k-1)!(n-k-1)!(b-a)^{n-1}} \times$$
$$\times \int_a^b (t-a)^{n-1}(b-t)b(t)^{n-1}\,dt. \tag{13}$$

5. Formula (13) may be used, in particular, to estimate from below the smallest, in modulus, eigenvalue of the problem (11)—(12), which we shall denote by $\lambda_1[q]$.

Lemma 2. The inequality

$$|\lambda_1[q]| \geq \lambda_1\big[(-1)^{q^{k+1}}|q|\big]$$

holds. This follows from the fact that $(-1)^k G(t,s) \geq 0$ and from the known inequality between the spectral radii of a kernel and of its modulus. Below, for brevity, we put

$$I[p] = \frac{1}{(n-1)(b-a)^{n-1}} \int_a^b (t-a)^{n-1}(b-t)^{n-1} p(t)\, dt.$$

Theorem 3. The estimate

$$|\lambda_1(q)| > \frac{(k-1)!(n-k-1)!}{I[\,|q|\,]} \tag{14}$$

is valid.

By virtue of Lemma 2 it is enough to restrict ourselves to the case of a real $q(t)$ such that $(-1)^{k+1}q(t) \geq 0$. But for this case one can make use of very deep results of M. G. Krein (1), which, in particular, show that the problem under consideration has an infinite number of eigenvalues $\lambda_1, \lambda_2, ...$, and that all of them are simple, real, and positive. The latter shows that $\lambda_1^{-1} < \sum_i \lambda_i^{-1}$, which, in view of (13), also completes the proof.

6. In what follows all functions are assumed real. Often estimates are needed only for real eigenvalues; at the same time it seems plausible that, for estimating from below the positive (or from above the negative) eigenvalues, $q(t)$ in (14) may be replaced by the functions $q_+(t) = \max\{0, q(t)\}$ or $q_-(t) = \max\{0, -q(t)\}$, depending on parity considerations. For a number of cases this proposition is confirmed. Namely, suppose that at least one of the relations $|n-2k| \leq 2,\ k = 1,\ k = n-1$ is fulfilled. If k is odd (even), then for the smallest positive eigenvalue of the problem (11)–(12), (14) is valid with $|q|$ replaced on the right-hand side by q_+ (q_-). For negative eigenvalues q_+ and q_-, obviously, should be interchanged. This proposition includes, in particular, all problems of the type under consideration for $n < 7$. The remaining cases require additional investigation.

Below we use the notation and terminology of the paper (6). If the positive eigenvalues of (11)–(12) are greater than 1, then $x^{(n)} + q(t)x = 0$ has no nontrivial solutions with $(a, n-k; b, k)$-zeros. Further, it is easy to verify that if $[a_1, b_1] \subset [a, b]$ and $p(t) \geq 0$, then $I_1[p] \leq I[p]$, where $I_1[p]$ is defined analogously to $I[p]$, but with a, b replaced by a_1, b_1. Therefore (14) can be rewritten in the form

$$I[\,|q|\,] > (k-1)!(n-k-1)! \to x^{(n)} + q(t)x \in T_{n-k,k}, \tag{15}$$

where, for the cases noted above, $|q|$ in the left-hand side of (15) can be replaced by q_+ for odd k and by q_- for even k.

Corollary 1. *If $I[q_+] \leq (n-2)!$, then Chaplygin' s theorem is valid for the operator $x^{(n)} + q(t)x$ on $[a, b]$.*

This follows immediately from the preceding, since $T_{n-1,1} \subset T$. What has been said also makes it possible to obtain an effective and at the same time sufficiently sharp condition for the inclusion $x^{(n)} + q(t)x \in T_0$. Recall that the nonoscillation of L entails a number of strong consequences (factorizability into "multipliers," various comparison theorems, applicability to $Lx + \lambda p(t)x = 0$ of the mentioned theorems of M. G. Krein, etc.).

Define the numbers α_n, β_n by the equalities ($n \geq 2,\ \beta_2 = \infty$)

$$\alpha_{2m+1} = \beta_{2m+1} = (m-1)!\, m!, \qquad \alpha_{4m+2} = [(2m)!]^2,$$

$$\alpha_{4m} = (2m-2)!(2m)!, \qquad \beta_{4m+2} = (2m-1)!(2m+1)!, \qquad \beta_{4m} = [(2m-1)!]^2.$$

Theorem 4. *If simultaneously $I[q_+] \leq \alpha_n$, $I[q_-] \leq \beta_n$, then the operator $x^{(n)} + q(t)x$ is nonoscillatory on $[a, b]$.*

Here, as in the preceding estimates, the constants and the weight function cannot be improved for each n. This can be explained as follows: if $q(t) = c\delta(t - t_0)$, then all the b.v.p.' s (11)–(12), except one, "go to infinity," so that for this limiting case equality holds in (14).

The last results for some special cases were obtained earlier by the author ([4],[6]) by other means; in this case the estimates were formulated in a less precise form, namely in terms of the smallness of the integrals of $q_+(t)$, $q_-(t)$, which are connected with $I[q_+]$, $I[q_-]$ by obvious inequalities. The method set forth above for estimating the smallest eigenvalue, whose main links are formula (10), the determination of $G(t, t)$, Lemma 2, and M. G. Krein' s theorem, can also be applied to a broader class of boundary-value problems.

7. In proving the following assertion, use is made, in particular, of certain facts from the theory of positive operators ([3]) (see also ([5])).

Theorem 5. *Let $L_0 \in T_0$, $q(t) \geq 0$. Then $Lx \equiv L_0x + (-1)^{k+1}q(t)x \in T_{n-k,k}$ if and only if there exists a $y(t)$ with $(a, n-k; b, k)$-zeros such that $y(t) > 0$, $(-1)^k Ly \geq 0$, $Ly \neq 0$ $(a < t < b)$.*

Corollary 2. *Let $L_0 \in T_0$, $q(t) \geq 0$ (≤ 0). Then $Lx \equiv L_0x + q(t)x \in T_0$ if and only if, for every odd (even) k, $1 \leq k \leq n-1$, there exists $y_k(t)$ with $(a, n-k; b, k)$-zeros such that, in the interval (a, b), $y_k(t) > 0$, $Ly_k \leq 0$ (≥ 0), $Ly_k \neq 0$.*

If L_0 is self-adjoint (for example, $L_0x = (x^{(m)})^{(m)} - (px^{(m-1)})^{(m-1)}$, $p(t) \geq 0$), then it suffices to consider only $k \leq n/2$. Therefore, for example, a criterion for the nonoscillation of the operator $x^{(8)} + q(t)x$ with sign-constant $q(t)$ is formulated in terms of the existence of only two functions ($y_1(t), y_3(t)$ for the case $q(t) \geq 0$, $y_2(t), y_4(t)$ for $q(t) \leq 0$).

Voronezh State
University

Received
8 V 1964

CITED LITERATURE

[1] M. G. Krein, *DAN*, **25**, No. 9 (1939).
[2] A. O. Gel' fand, Appendix to the book by U. V. Lovitt, *Linear Integral Equations*, 1957.
[3] M. A. Krasnosel' skii, *Positive Solutions of Operator Equations*, Moscow, 1962.
[4] A. Yu. Levin, *DAN*, **148**, No. 3 (1963).
[5] N. V. Azbelev, Z. B. Tsalyuk, *DAN*, **156**, No. 2 (1964).
[6] A. Yu. Levin, *DAN*, **156**, No. 6 (1964).

Note: Figure translations are in progress. See original paper for figures.