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Abstract

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On the Spectrum of Non-Self-Adjoint Differential Operators

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In the space $L_2(E_n)$ we consider a closed differential operator T, which arises as the result of perturbing a self-adjoint differential operator A by a differential operator B of lower order. The structure of the spectrum of such operators was studied in papers ($^{1-3}$). However, in these papers only the case was considered in which the perturbation B is the operator of multiplication by a bounded continuous function tending to zero at infinity. In the present article the structure of the spectrum is investigated for the case of unbounded perturbations of certain elliptic self-adjoint operators.

In what follows, the principal role is played by the following

Theorem 1. Let a self-adjoint operator A with domain $D(A)$ be given in a Hilbert space $\mathfrak{H}$, and let the operator B satisfy the following conditions:

1. $D(B) \supset D(A)$.
2. There exist constant real numbers a and b, $0 \le a < {}^1/_3$, $b > 0$, such that for any $f \in D(A)$
$$\|Bf\| \le a\|Af\| + b\|f\|.$$
3. There exists an integer m such that the operator
$$\left(B \cdot \frac{1}{A - z}\right)^m$$
is completely continuous at every point z belonging to the domain G.
4. At least at one point $z_0 \in G$ the operator
$$E + B \cdot \frac{1}{A - z}$$
has a bounded inverse.

Then:

1. The operator $T = A + B$, defined on the set $D(A)$, is closed.
2. The points of the spectrum of the operator T lying in the domain G are isolated eigenvalues of finite multiplicity.

We outline the proof of the theorem. From conditions 1 and 2 it follows that, in any case for points $z = \pm i\tau$, $\tau > 0$, with τ sufficiently large, there exists and is bounded the operator $(T \pm i\tau)^{-1}$, mapping $\mathfrak{H}$ onto $D(A)$. Hence it follows that the operator T is closed.

Next it is easy to see that for those z for which the bounded operator

$$\left(E + B \cdot \frac{1}{A - z}\right)^{-1}$$

exists, the identity

$$(T - z)^{-1} = (A - z)^{-1} \left(E + B \cdot \frac{1}{A - z}\right)^{-1}$$

holds.

Fix a point $z_0 \in G$. By virtue of condition 3, using a theorem of S. M. Nikol' skii (^{4}), we conclude that z_0 belongs to the Fredholm domain for the operator

$$K_z = B \cdot \frac{1}{A - z},$$

and, moreover, in the domain G the operator K_z depends analytically on z. Consequently, one may apply a theorem of I. Ts. Gokhberg (^{5}). Taking into account condition 4 of the theorem, we exclude the case in which all points of the domain G are eigenvalues of the operator T. This completes the proof of the theorem.

The following follows immediately from Theorem 1.

Theorem 2. Suppose that, in the conditions of Theorem 1, the integer m is equal to one; then:

1. *The non-real spectrum of the operator T is a set of isolated eigenvalues of finite multiplicity.*
2. *The points of the continuous spectrum of the operator A are points of the spectrum of the operator T.*

The first assertion of the theorem in the case where B is a bounded operator was first established in the work of I. M. Gelfand (^{1}) (see also (^{6,7}) in this connection). Therefore we shall prove only the second assertion. Suppose that a point λ—a point of the continuous spectrum of the self-adjoint operator A—is a regular point of the operator T. Let $g \in \mathcal{H}$. Consider the equation

$$f - B \cdot \frac{1}{T - \lambda} f = g; \tag{1}$$

it is not difficult to show that the operator $B \cdot \frac{1}{T - \lambda}$ is completely continuous and that the homogeneous equation

$$f - B \cdot \frac{1}{T-\lambda} f = 0$$

has no solutions different from zero. Therefore equation (1) is uniquely solvable for every $g \in \mathcal{H}$. Hence there follows the unique solvability of the equation

$$(A - \lambda E)\varphi = g,$$

which contradicts the assumption that λ is a point of the spectrum of the self-adjoint operator.

Let us consider applications of Theorems 1 and 2 to the case of differential operators.

Theorem 3. *Let a differential operator A be given in $L_2(E)$, and let $Tu = Au + q(x)u$, where A is a self-adjoint differential operator whose resolvent is an integral operator with kernel $K(x, y)$ such that, for some integer m, the function*

$$A(x, g) = \int_{E_n} |K(x, t_1) \dots K(t_m, y)| \, dt_1 \dots dt_m$$

satisfies the condition

$$\int_{E_n} A^2(x, y) \, dy \le C(x),$$

where the constant $C(x)$ is uniformly bounded for all x varying in each compact set, and the function $q(x)$ is a continuous complex-valued function satisfying the condition

$$\lim |q(x)| = 0, \quad |x| \to \infty.$$

Then the non-real spectrum of the operator T consists of isolated eigenvalues of finite multiplicity.

Corollary 1. *Let A be an elliptic self-adjoint differential operator of order $2k$, with $k > n/4$; then, as was shown by L. Gårding (8), one may take m equal to one; consequently, for such operators Theorem 3 is valid.*

The case of the operator

$$Au = -\Delta u + q(x)u + ip(x)u$$

was first considered by V. B. Lidskii (2).

Corollary 2. *Let A be an elliptic self-adjoint differential operator bounded below, with sufficiently smooth bounded coefficients. A. G. Kostyuchenko* (^{9}) *proved that in this case there exists an integer m satisfying condition 2 of Theorem 1. Hence, in this case Theorem 3 is also valid.*

We note that this result generalizes the corresponding theorem of R. M. Martirosyan (^{3}), who considered the analogous case of the operator A, but with constant coefficients.

Theorem 4. Let

$$Af = \left(- \sum \frac{\partial^2}{\partial x_j^2} \right)^l f$$

with domain $D(A)$:

$$D(A) = \left\{ f, \ f = \int_{E_n} \frac{\varphi(p) e^{i(p,x)}}{|p|^{2l} + 1} \, dp, \ \varphi(p) \in L_2(E_n) \right\}.$$

On the functions of the set $D(A)$ define the differential operator

$$Bf = \sum_{|\alpha| \le 2l-1} a_\beta(x) \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} f(x),$$

where $a_\beta(x)$ are continuous complex-valued functions such that $|a_\beta(x)| \to 0$ as $|x| \to \infty$.

Then:

1. The operator $T = A + B$, defined on the set $D(A)$, is closed.
2. The spectrum of the operator T consists of the half-axis $\lambda \ge 0$ and of at most a countable set of isolated eigenvalues of finite multiplicity.

To prove Theorem 4 we verify that all the conditions of Theorem 2 are satisfied for the operators A and B.

Using the condition of the theorem on the functions $a_\beta(x)$ and passing to the Fourier transform of the function $f(x)$, it is not difficult to show that for functions of the form $Bf, \ f \in D(A)$, the estimate

$$\|Bf\| \le a\|Af\| + b\|f\|$$

is valid, where the constant a can be chosen arbitrarily small.

Next we show that the operator

$$B \cdot \frac{1}{A-z}$$

is completely continuous for every z not belonging to the positive part of the real axis. Consider the operator

$$B_\beta f = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} \frac{1}{A-z} f \qquad (|\beta| \leq 2l-1),$$

and let the elements f satisfy the condition $\|f\| \leq C$. Then it is not difficult to show that

$$\|B_\beta f\| \leq C_1, \qquad \left\| \frac{\partial}{\partial x_j} B_\beta f \right\| \leq C_2$$

uniformly in f. Therefore, for the family of functions $u = B_\beta f$, $\|f\| \leq C$, considered only inside the ball K_L of radius L with center at the origin, the inequalities

$$\|B_\beta f\|_{L_2(K_L)} \leq C_1, \qquad \|\operatorname{grad} B_\beta f\|_{L_2(K_L)} \leq C_2$$

hold.

Therefore the family of functions $u = B_\beta f$ is compact on the ball K_L in the sense of $L_2(K_L)$. Choose now L so that outside the ball K_L, $|a_i(x)| < \varepsilon$, and let $\varphi_1, \dots, \varphi_t$ be a finite ε-net for the set $B_\beta f$, compact in $L_2(K_L)$. Then, if the functions $\varphi_1, \dots, \varphi_t$ are extended by zero to the whole space E_n, it is not difficult to show that the functions $\varphi_1, \dots, \varphi_t$ so obtained will be a finite $C\varepsilon$-net for the set of functions $a_\beta(x) B_\beta f$ in the sense of $L_2(E_n)$ (here C does not depend on ε). By the arbitrariness of ε this proves the compactness of the operator $a_\beta B_\beta$ for any β in the sense of $L_2(E_n)$, $|\beta| \leq 2l-1$. Hence the compactness in $L_2(E_n)$ also follows for the operator

$$B \cdot \frac{1}{A-z}.$$

Thus we have reduced Theorem 4 to Theorem 2.

Finally, let us note that Theorem 1 admits the following generalization to the case when A is a normal operator.

Theorem 5. Let $H_1, \dots, H_k$ be a family of pairwise commuting self-adjoint operators, and let $F(t)$ be a measurable, almost everywhere finite, complex-valued function of k real variables $t_1, \dots, t_k$, mapping the set $\{M_1 \dots M_k\}$, where M_j is the spectrum of the operator H_j, onto a connected set M in the complex z-plane.

Consider the operator

$$F(H_1, \dots, H_k) = F(H)$$

and suppose that:

1. There exists a straight line $\mathcal{L}$, whose intersection with the set M is a closed segment l, at least one of whose ends is situated in the finite part of the plane, and for points $z,\ z \in \mathcal{L} - l$, the inequality

   $$\left\| \frac{1}{F(H) - z} \right\| \leq \frac{1}{d(z, l)}$$

   holds, where $d(z, l)$ is the distance from the point z to the segment l.

2. In addition, suppose that an operator V is given such that: a) $D(V) \supset D(F(H))$; b) there exist constants $b > 0,\ 0 \leq a < 1/3$, such that for every $f \in D(F(H))$

   $$\|Vf\| \leq a\|F(H)f\| + b\|f\|;$$

c) for some $z_0,\ z_0 \notin M$, the operator

   $$V \frac{1}{F(H) - z}$$

   is completely continuous.

Then:

1. The operator

   $$T = F(H) + V,$$

   defined on $D(F(H))$, is closed.

2. The spectrum of the operator T not belonging to M consists only of isolated eigenvalues of finite multiplicity.

3. The points of the continuous spectrum of the operator $F(H)$ are spectral points of T.

The proof of this theorem is carried out in the same way as the proof of Theorem 1.

As an example, consider the operator H with domain of definition $D(H)$:

$$Hf = \left(-\sum \frac{\partial^2}{\partial x_j^2} \right)^t + \sum_{l=1}^{t} \sum_j a^{2l-1}_{j_1 \cdots j_k} \frac{\partial^{2l-1} f}{\partial x_1^{j_1} \dots \partial x_n^{j_n}},$$

$$D(H) = \left\{ f,\ f = \int \frac{\varphi(p) e^{i(p,x)}}{p^{2t} + 1}\, dp,\ \varphi(p) \in L_2(E_n) \right\}.$$

Here $a^{2l-1}_{j_1 \cdots j_k},\ |j| \leq 2l - 1$, are real numbers.

On $D(H)$ define the operator V

$$Vf = \sum_{|\alpha| \le m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} f(x),$$

where $2t - 2(m+1) - n > 0$ and $a_\alpha(x)$ are continuous complex-valued functions satisfying the conditions: 1) $a_\alpha(x) \in L_2(E_n)$; 2) $|a_\alpha(x)| \to 0$ as $|x| \to \infty$.

Then, for the operators H and V so defined, all the conditions of Theorem 5 are fulfilled.

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References

1. I. M. Gelfand, UMN, 7, 6, 183 (1952).
2. V. B. Lidskii, DAN, 112, No. 6, 994 (1957).
3. R. M. Martirosyan, Izv. AN SSSR, ser. matem., 27, 3, 677 (1963).
4. S. M. Nikolskii, DAN, 11, No. 8, 309 (1936).
5. I. Ts. Gokhberg, DAN, 78, No. 4, 629 (1951).
6. I. Ts. Gokhberg, M. G. Krein, UMN, 12, 2 (74), 43 (1957).
7. Z. I. Khalilov, Uch. zap. Azerb. univ., 1, 15 (1955).
8. L. Gordin, Sborn. per. Matematika, 1, 3, 113 (1957).
9. A. G. Kostyuchenko, DAN, 115, No. 1, 34 (1957).

Note: Figure translations are in progress. See original paper for figures.

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