AN EVOLUTIONARY METHOD FOR SOLVING PROBLEMS OF ELECTRODYNAMICS

1964

SovietRxiv

Abstract

Full Text

MATHEMATICAL PHYSICS

S. P. LOMNEV

AN EVOLUTIONARY METHOD FOR SOLVING PROBLEMS OF ELECTRODYNAMICS

(Presented by Academician A. A. Dorodnitsyn, 14 IV 1964)

The development of electronic computing machines will make it possible to approach the solution of the fundamental problem (1) of classical electrodynamics: finding the distribution of the field and of particles in time and space, for prescribed initial and boundary conditions and a prescribed initial distribution of charges and currents, from the complete system of Maxwell equations.

Below it is proposed to solve this problem by an evolutionary method: by steps in t and by grids in space. The problem for $\varepsilon = \mu = 1$ is reduced to the system of equations

$$\Delta\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 4\pi\left(\frac{1}{c^2}\frac{\partial\mathbf{j}}{\partial t} + \operatorname{grad}\rho\right),$$

$$\Delta\mathbf{H} - \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = -\frac{4\pi}{c}\operatorname{rot}\mathbf{j} \qquad (1)$$

with prescribed initial values

$$\mathbf{E}(q_1, q_2, q_3, 0) = \mathbf{E}_0,$$

$$\mathbf{H}(q_1, q_2, q_3, 0) = \mathbf{H}_0, \qquad (2)$$

boundary conditions of the form

$$\frac{\partial\mathbf{E}}{\partial n} + \sigma_E\mathbf{E} = 0; \qquad \frac{\partial\mathbf{H}}{\partial n} + \sigma_H\mathbf{H} = 0, \qquad (3)$$

and ρ and $\mathbf{j}$ are constructed from the solution of the equation of motion

$$\left(\frac{d\mathbf{p}_m}{dt} = e_m\left\{\mathbf{E}_m + \frac{1}{c}\left[\dot{\mathbf{R}}_m\mathbf{H}_m\right]\right\}\right)_{m=1}^{M}; \qquad (4)$$

$\mathbf{p}_m$ is the momentum vector of a particle; $\dot{\mathbf{R}}_m$ is its velocity; $\mathbf{E}_m,\ \mathbf{H}_m$ are the field, determined from (1) for many "coarsened charges" or "observation points," with

$$\rho(q_1, q_2, q_3, t_p) = \sum \alpha_\nu n_\nu(q_1, q_2, q_3, t_p),$$

$$\mathbf{j}(q_1, q_2, q_3, t_p) = \sum \rho_\nu(q_1, q_2, q_3, t_p)\dot{\mathbf{R}}_{i,j,k,p,\nu}; \tag{4a}$$

$n_\nu(q_1, q_2, q_3, t_p)$ is the number of "coarsened particles" of species ν with charge q_ν in a unit volume $\Delta V_{i,j,k}$ at time p; $\dot{\mathbf{R}}_{ijk\nu}$ is the mean velocity of the particles of this volume, and α_ν is the density measure in the volume i, j, k; $\alpha_\nu = q_\nu/\Delta V_{i,j,k}$.

Let us write (1) in difference form with indices of the spatial grid i, j, k and p, the number of the time step. Knowing the solution of (1) at the p-th instant of time, from the equation of motion we compute $\rho_{i,j,k,p+1}$ and $\mathbf{j}_{i,j,k,p+1}$ and, again finding $\mathbf{E}$ and $\mathbf{H}$ from (1), determine $\rho_{i,j,k,p+2}$ and $\mathbf{j}_{i,j,k,p+2}$, etc. The difference scheme, the grid size $\Delta q_1, \Delta q_2, \Delta q_3$, and the magnitude of the time step Δt_p are found with the aid of experimental computation and, generally speaking, their relation is different for each problem.

Equation (4) is reduced to the Cauchy problem for a system of ordinary differential equations on the interval Δt_p and is integrated numerically. The $\mathbf{E}$ and $\mathbf{H}$ entering on the right are functions only of the coordinates and, at any point, are found by interpolation over neighboring nodes of the spatial grid.

At present, because of the limited memory and speed of electronic computing machines, it is generally not possible to obtain an exact solution by directly integrating system (1)–(4). It is necessary, on the basis of a preliminary analysis of the elementary phenomena, taking into account the features of the process and possible coarsenings, gradually to complicate the problem and obtain the main part of the solution of the evolutionary problem according to the following scheme:

A. The electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ are determined, approximately or from experiment, for zero density and current of particles $\rho = j = 0$. This stage corresponds to the formulation of the problem.

B. In the field thus obtained, the trajectories of individual particles are found, and the parameters and forms of the boundaries that give the desired form of these trajectories are selected.

C. A rigorous solution of (1) is carried out for $\rho = 0,\ j = 0$ and for the chosen field structure from B.

D. On the basis of the field computed in C, the behavior of "coarsened charges" [3] or "observation points" [2], which describe the behavior of all particles, is considered. The field of the particles themselves is taken in the form of retarded potentials.

Fig. 1. Diagram of the transverse section of a waveguide cell

Figure 1: Fig. 1. Diagram of the transverse section of a waveguide cell

Fig. 1. Diagram of the transverse section of a waveguide cell

E. Finally, problem D is solved with the boundary conditions taken into account, i.e., the system closely approximating the complete system of Maxwell equations.

Each of these stages helps to isolate the main features of the phenomenon and usually greatly simplifies the solution, without appreciably reducing its accuracy. Therefore the final form of the system for each particular problem is much simpler than (1)–(4) and, as a rule, is amenable to machine processing.

As an example, let us consider the motion of a beam of particles in a cell of a diaphragm-loaded waveguide. A preliminary analysis according to A–D [3] makes it possible to formulate the problem in the following form:

$$\Delta E_z - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 4\pi\left[\frac{\partial j(z,t)}{c^2\partial t} + \frac{\partial \rho(z,t)}{\partial z}\right]; \tag{1'}$$

with

$$E_z = 0 \quad \text{on } II \text{ and } IV \text{ (Fig. 1);} \tag{5a}$$

$$\frac{\partial E_z}{\partial z} = 0 \quad \text{on } III \tag{5b}$$

and the periodicity condition on I

$$E_z\left(-\frac{D}{2}, r\right) = e^{i\beta_0 D} E_z\left(\frac{D}{2}, r\right). \tag{5c}$$

As the initial values one takes the solution of $(1')$ $(\rho = j = 0)$

$$\Delta E_z + k^2 E_z = 0; \qquad k = \omega/c \tag{6}$$

with boundary conditions (3), obtained, for example, by the grid method [4].

The periodicity condition for $t > 0$ is not rigorous, since Floquet' s theorem is not valid and on I the boundary condition does not depend harmonically on time. This means that the problem will not take into account at least the losses of the high-frequency field due to particle acceleration. The inaccuracy of the formulation for the real case can be corrected in the following way.

Write the solution of $(1')$ in the form $E_z = E_{z0} + \widetilde{E}_z$, where E_{z0} is the solution of (6) with boundary conditions (5a), (5b), (5c), and $\widetilde{E}_z$ is the solution of (1) with zero initial values, boundary conditions (5a), (5b), and the condition

$$\widetilde{E}_z(r)\Big|_{\pm D/2} = \sum_{l=1}^{L} \frac{q_l(\pm D/2 - z_l)}{\left[(\pm D/2 - z_l)^2 + r^2\right]^{3/2}}. \tag{5d}$$

(where l is the index of the enlarged charge q_l, z_l is its coordinate), corresponding to free radiation through the boundary, $0 \leqslant r \leqslant a;\ \ z = \pm D/2$.

$(1')$ was written in the following difference form:

$$\Delta E_{zi,j}\Big|_p = \frac{E_{i+1,j} - 2E_{i,j} + E_{i-1,j}}{\Delta z^2} + \frac{E_{i,j+1} - 2E_{i,j} + E_{i,j-1}}{\Delta r^2} + \frac{1}{r_{ij}} \frac{E_{i,j+1} - E_{i,j-1}}{2\Delta r}, \tag{1”}$$

$$\frac{\partial^2 E_z}{\partial t^2}\Bigg|_{i,j} = \frac{E_{z,p+1} - 2E_{z,p} + E_{z,p-1}}{\Delta t_p^2};$$

ρ and j at time $t = 0$ were set equal to 0, and at $t = \Delta t_p$ were determined from the solution of

$$\ddot{z}_l = A_0(1 - \dot{z}_l^2)^{3/2} E_{z,l}, \tag{7}$$

$$\rho(z_i, t) = \alpha n(z_i, t), \tag{8}$$

where α is a certain constant determined by the initial charge density; $n(z_i, t)$ is the number of observation points on the segment Δz_i with the value E_z at $t = 0$.

As initial values for (7) there were taken

$$z_{0l} = -D/2 \quad \text{at } t = \Delta t_p l; \qquad l = 0, 1, \ldots, L; \qquad \dot{z}_{0l} = \text{const}. \tag{9}$$

In the case

$$-D/2 \geqslant z_l(t); \qquad z_l(t) \geqslant D/2; \tag{10}$$

the equation with such l was discarded.

The formulation of the problem in the form (1), (3), (5), (7) and (8), (9) corresponded to the natural character of the course of the process—the passage of particles through the waveguide cell. Condition (10) meant the absence of coupling between the charges of neighboring cells, although in principle this restriction can be avoided.

Trial calculations were performed for the motion of a beam of particles in certain types of cells at a wave phase velocity $\simeq 1$. The results are as follows:

1) $\varphi_{\text{cp}\, j\neq 0} = (\varphi_1 + \varphi_2)/2$ is shifted in comparison with the case $j = 0$ (here φ_1 and φ_2 are the phases of the extreme particles). For example, for a field on the axis with amplitude $E_{0\,\text{max}} = 30$ kV/cm,

$$\varphi_{\text{cp}\, j=1a} - \varphi_{\text{cp}\, j=0} \simeq 0.001 \text{ rad/cm},$$

$$\varphi_{\text{cp}\, j=10a} - \varphi_{\text{cp}\, j=0} \simeq 0.01 \text{ rad/cm}.$$

For $E_{0\,\text{max}} = 100$ kV/cm,

$$\varphi_{\text{cp}\, j=10a} - \varphi_{\text{cp}\, j=0} \simeq 0.003 \text{ rad/cm}.$$

2) $\Delta\varphi = \varphi_1 - \varphi_2$ increases.

In the case $E_{0\,\text{max}} = 30$ kV/cm,

$$\Delta\varphi_{j=1a} - \Delta\varphi_{j=0} \simeq 0.00075 \text{ rad/cm},$$

$$\Delta\varphi_{j=10a} - \Delta\varphi_{j=0} \simeq 0.0075 \text{ rad/cm}.$$

For $E_{0\,\text{max}} = 100$ kV/cm,

$$\Delta\varphi_{j=10a} - \Delta\varphi_{j=0} \simeq 0.0025 \text{ rad/cm}.$$

The figures given in 1) and 2) show the necessity of taking space charge into account in large accelerators (of length more than 1 m) at large currents (more than 1 A).

Computing Center
Academy of Sciences of the USSR

Received
9 IV 1964

CITED LITERATURE

1. I. E. Tamm, *Foundations of the Theory of Electricity*, Moscow, 1956.
2. S. P. Lomnev, DAN, 143, No. 6 (1962).
3. S. P. Lomnev, *Methods for Calculating Linear Electron Accelerators*, Publishing House of the Academy of Sciences of the USSR, 1962.
4. N. P. Zhidkov, I. S. Berezin, *Methods of Computation*, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.