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MECHANICS

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Abstract

Full Text

MECHANICS

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ON THE STABILITY OF EQUILIBRIUM OF SYSTEMS WITH AN INFINITE NUMBER OF DEGREES OF FREEDOM IN THE SENSE OF LYAPUNOV

(Presented by Academician M. A. Lavrent'ev on February 8, 1964)

I. To demonstrate a number of specific features inherent in the concept of stability of equilibrium of systems with an infinite number of degrees of freedom, we have investigated the character of the stability of the trivial equilibrium position of three linear elastic systems: a string fixed at its ends, a rectangular membrane fixed along its contour, and an elastic isotropic body in the form of a rectangular parallelepiped, rigidly clamped over its entire surface (active external forces are absent in all three cases). By means of d' Alembert's formulas ($\hat{1}$), Poisson's formulas ($\hat{2}$), and Stokes-Ostrogradsky's formulas ($\hat{3}$), for the indicated systems the solution of the Cauchy problem is constructed by the method of reflection. As a result of the analysis of these solutions* the following conclusions** are obtained:

1. In the motion caused by an arbitrary initial disturbance, the displacement of each point of the system from the equilibrium position is arbitrarily small if, at the initial instant of time, sufficiently small are: for the string—the displacement and the velocity; and for the membrane or the elastic body, in addition, the gradient of the displacement.
2. For the membrane or the elastic body, smallness of the initial displacements and velocities does not ensure smallness of the subsequent displacements even on an arbitrarily small time interval.
3. In the motion caused by an arbitrary initial disturbance, the displacement, the gradient of the displacement, and the velocity are arbitrarily small if, at the initial instant of time, sufficiently small are: for the string—the displacement, the gradient of the displacement, and the velocity; and for the membrane or the elastic body, in addition, the second derivatives of the displacement with respect to the coordinates and the gradient of the velocity.
4. For the string, smallness of the initial displacements and velocities does not ensure, in the subsequent motion, smallness of the velocities even on

an arbitrarily small time interval.

For the membrane and the elastic body, smallness of the initial values of the displacement, the gradient of the displacement, the velocity, and the gradient of the velocity, or smallness of the initial values of the displacement and of its derivatives with respect to the coordinates of the first and second orders and of the velocity, does not ensure, in the subsequent motion, smallness of the velocities even on an arbitrarily small time interval.

II. Let some mechanical system with an infinite number of degrees of freedom be in a state of equilibrium^{***}. A judgment about the stability of this equilibrium must be based on comparison with it of other motions of the system possible for it under the same forces and constraints that were taken into account in finding the equilibrium, i.e., on comparison of various solutions of the equations of motion of the system with the time-independent solution describing the equilibrium. The equations of motion in this case will be

* The investigation is carried out in the class of solutions twice continuously differentiable up to the boundary and existing on an infinite time interval.

** The results for the string are established on an infinite time interval; for the membrane and the elastic body, on an arbitrary but fixed finite time interval.

*** Systems with delay, as well as systems with a countable number of degrees of freedom, are excluded from consideration.

some partial differential equations or functional equations of a more general form, boundary conditions and, possibly, conditions at infinity, if the system occupies an unbounded region of space. The role of the initial values of the independent coordinates and velocities of a system with a finite number of degrees of freedom is played, for our system, by the Cauchy initial data. Let D be some class of solutions of the equations of motion*. We denote by $M_{t_0}(D)$ the class of Cauchy initial data defined by the following property: the initial data belong to the class $M_{t_0}(D)$ if and only if there exists a solution of the equations of motion of class D satisfying these initial data at $t = t_0$. Combining the classes $M_{t_0}(D)$ corresponding to all possible initial instants of time t_0 , we obtain a set $M(D)$, which we shall call the **state space of the system**. For our system, $M(D)$ plays a role analogous to that of the phase space of the independent coordinates and velocities of a system with a finite number of degrees of freedom. To every motion of the system of class D there corresponds a motion $m(t)$ of the representative point in $M(D)$. In particular, the equilibrium position under consideration is represented in $M(D)$ by a certain fixed point m^0 .

Definition 1. By a **family of neighborhoods of the point m^0** we shall mean any nonempty collection $\{U\}$ of subsets U of the state space, if: 1) every set U of the collection $\{U\}$ contains the point m^0 ; 2) the intersection of any two sets U_1 and U_2 of the collection $\{U\}$ contains some set U_{12} of this collection.

The sets $U \in \{U\}$ will be called **neighborhoods of the point m^0 of the family $\{U\}$** .

Definition 2. Let $\{X\}$ and $\{Y\}$ be two families of neighborhoods of the point m^0 such that every neighborhood Y of the family $\{Y\}$ contains some neighborhood X of the family $\{X\}$. The equilibrium m^0 is called **stable in the class of motions D with respect to the families of neighborhoods $\{X\}, \{Y\}$** , if for every neighborhood $Y \in \{Y\}$ and every t_0 one can indicate a neighborhood $X \in \{X\}$ (depending on Y and t_0) such that, for every motion $m(t)$ of class D for which $m(t_0) \in X$, one has $m(t) \in Y$ for every $t \geq t_0$ from the time interval on which this motion exists.

The definition just given indicates the general structure of a broad class of definitions of stability constructed in the spirit of Lyapunov's definition⁽⁴⁾.

Definition 3. We shall say that a definition of stability of the equilibrium m^0 with respect to the families of neighborhoods $\{X\}, \{Y\}$ is **incorrect** if, in the sense of this definition, the equilibrium m^0 is unstable on an arbitrarily small time interval. An example of such a definition is the definition of stability of equilibrium "with respect to coordinates and velocities" for a string, a membrane, or an elastic body (Sec. I). A number of paradoxical facts concerning the instability of equilibrium of systems with an infinite number of degrees of freedom, discovered recently^(5,6), are likewise connected with the incorrectness of the corresponding definitions of stability. It is obvious that any reasonable definition of stability must be correct.

Correct definitions of stability of the equilibrium of a system with an infinite number of degrees of freedom may have an entirely different content than in the case of systems with a finite number of degrees of freedom. The fact that a given definition of stability is correct is equivalent to a certain theorem on the continuous dependence of the solution of the equations of motion of the system on the initial data. Thus the concept of stability of equilibrium of systems with an infinite number of degrees of freedom is closely connected with the fundamental questions of the existence of solutions and the correctness of the formulation of the Cauchy problem for the equations of perturbed motion.

III. An analogue of the theorem of Lyapunov's direct method on stability has been constructed, more convenient in applications in comparison with those proposed earlier

* It is assumed that the equilibrium under consideration belongs to the class D .^(7,8) Let Z be a nonempty set of elements z , T a fixed infinite time interval t unbounded on the right, and D a fixed set of mappings $z(t)$ of time intervals contained in T into the set Z . It is assumed that D contains the special mapping $z(t) \equiv z^0, t \in T$.

The mappings $z(t)$ belonging to D are called **motions**, and $z(t) \equiv z^0$ an **equilibrium**.

A **measure of perturbation** ⁽⁸⁾ is a real functional $\rho(z)$, defined everywhere on Z , such that: a) $\rho(z) \geq 0$, b) $\rho(z^0) = 0$.

Let $\rho_1 \dots \rho_s$ be a finite collection of measures of perturbation and let $\varepsilon_1 \dots \varepsilon_s$ be arbitrary positive numbers. An $\varepsilon_1 \dots \varepsilon_s$ -**neighborhood of the point z^0 with respect to the system of measures $\rho_1 \dots \rho_s$** is the set of points $z \in Z$ for which $\rho_i(z) < \varepsilon_i$ for every i , $1 \leq i \leq s$.

Obviously, the collection of all possible $\varepsilon_1 \dots \varepsilon_s$ -neighborhoods of the point z^0 with respect to the system of measures $\rho_1 \dots \rho_s$ forms a family of neighborhoods of this point in the sense of the definition in Sec. II, item 1.

Let $V(z, t)$ be a real functional defined on $Z \times T$.

- a) $V(z, t)$ is called **upper semicontinuous with respect to the system of measures $\rho_1 \dots \rho_s$** , if for any $\varepsilon > 0$ one can specify a neighborhood of the point z^0 with respect to the system of measures $\rho_1 \dots \rho_s$ such that $V(z, t) < \varepsilon$ for any point z of this neighborhood and any $t \in T$.
- b) $V(z, t)$ is called **positive definite** ⁽⁹⁾ with respect to the system of measures $\rho_1 \dots \rho_s$ if the lower bound of $V(z, t)$ on the boundary* of any sufficiently small neighborhood of the point z^0 with respect to the system of measures $\rho_1 \dots \rho_s$ is a strictly positive number (uniformly in $t \in T$).
- c) $V(z, t)$ is called **nonincreasing along motions** if, for any $z(t) \in D$, the function $V(z(t), t)$ does not increase.

Theorem. Let $\rho_{01} \dots \rho_{0n}$ and $\rho_1 \dots \rho_m$ be two systems of measures of perturbation such that: a) every neighborhood of the point z^0 with respect to the system of measures $\rho_1 \dots \rho_m$ contains a neighborhood of this point with respect to the system of measures $\rho_{01}, \dots, \rho_{0n}$; b) every motion $z(t) \in D$ is continuous in each of the measures $\rho_1 \dots \rho_m$.

If there exists a functional $V(z, t)$: a) upper semicontinuous with respect to the system of measures $\rho_{01} \dots \rho_{0n}$; b) positive definite with respect to the system of measures $\rho_1 \dots \rho_m$; c) nonincreasing along motions, then the equilibrium z^0 is stable with respect to the two systems of measures $\rho_{01} \dots \rho_{0n}$ and $\rho_1 \dots \rho_m$.

- IV. With the aid of the theorem proved, for a nonlinear one-dimensional conservative system whose potential energy is a functional of the simplest problem of the calculus of variations, a justification is given of the energy criterion for stability of equilibrium in the following two cases: a) the potential energy of the system is positively regular in some extended strong neighborhood of equilibrium; b) the potential energy of the system is strongly positively regular in some strong neighborhood of equilibrium.

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* The boundary of a neighborhood of the point z^0 with respect to the measures $\rho_1 \dots \rho_s$ is defined in the natural way.

Note: Figure translations are in progress. See original paper for figures.

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