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Abstract

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On Some Criteria for the Existence of Periodic Solutions of Ordinary Differential Equations

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In papers ($^{1-3}$) the method of guiding functions for proving the existence of periodic solutions of systems of ordinary differential equations

$$\dot{x}_i = f_i(t, x_1, \dots, x_m) \qquad (i = 1, \dots, m) \tag{1}$$

with an ω-periodic right-hand side was proposed and developed. As is well known, the question of periodic solutions is easily reduced to the question of the existence of fixed points of the operator U of shift by the time $0 \leq t \leq \omega$ along the trajectories either of the system (1) itself or of some auxiliary system. To prove the existence of fixed points of the operator U, it is natural to use various topological characteristics of the mapping U. The method of guiding functions makes it possible to compute or estimate some of these topological characteristics if the right-hand sides of the system satisfy certain inequalities. The method of guiding functions is also applicable to proving the existence of solutions of system (1) bounded on the whole axis $-\infty < t < \infty$, with right-hand sides not possessing the property of periodicity; the theorems thereby obtained are close to those which follow from the well-known Ważewski principle (^{4}).

In this article we formulate criteria for the existence of periodic and bounded solutions, obtained mainly by the method of guiding functions. Some of these criteria are strengthenings of assertions from (^{1},^{2}) and of assertions obtained by A. I. Perov in his candidate dissertation (see (^{3})).

Below, by x we denote points of the m-dimensional space R^m. System (1) is written in vector form as

$$\dot{x} = f(t, x). \tag{2}$$

1. A continuously differentiable function $\Phi(x)$ has, by definition, index γ if its gradient does not vanish outside some ball $\|x\| \leq \rho_0$ and if the rotation (see (^{5},^{6})) of the vector field of its gradients on the spheres $\|x\| = \rho$ for $\rho = \rho_0$ is equal to γ. We recall that the index of even functions is odd.

The function $\Phi(x)$ is called guiding for system (2) if

$$(\operatorname{grad} \Phi(x), f(t, x)) > 0 \qquad (\|x\| \geq \rho_0). \tag{3}$$

Theorem 1. *Suppose that, for system (2), a guiding function of nonzero index (for example, an even one) can be specified. Suppose that every solution of the problem*

$$\dot{x} = f(t, x), \qquad x(0) = x_0, \tag{4}$$

for every $x_0 \in R^m$ is defined on the interval $[0, \omega]$. Then system (2) has at least one ω-periodic solution.

2. The investigation of system (2) becomes substantially more complicated if some solutions can "go to infinity" in a finite time. The existence of a single directing function is then no longer sufficient.

Here and below T denotes the set of those $x \in R^m$ such that $\|x\| \geq \rho_1$, where $\rho_1 > \rho_0$, and $m \leq \Phi(x) \leq M$, where m and M are, respectively, the least and greatest values of the directing function $\Phi(x)$ on the ball $\|x\| \leq \rho_0$.

Theorem 2. Let $\Phi(x)$ be a directing function for system (2) with nonzero index (for example, let $\Phi(x)$ be even). Suppose that on T there are given continuously differentiable functions $\Psi_1(x), \dots, \Psi_r(x)$ such that

$$(\operatorname{grad} \Psi_i(x), f(t, x)) \geq 0 \qquad (x \in T;\ i = 1, \dots, m), \tag{5}$$

$$\lim_{x \in T;\ \|x\| \to \infty} \{|\Psi_1(x)| + \dots + |\Psi_r(x)|\} = \infty. \tag{6}$$

Then system (2) has at least one solution defined on the whole axis and uniformly bounded.

Theorem 3. Suppose that the right-hand side of system (2) is ω-periodic in t and satisfies the conditions of theorem (2). Then system (2) has at least one ω-periodic solution.

3. The theorems of the preceding sections can be strengthened if a qualified lower estimate of the scalar product $(\operatorname{grad} \Phi(x), f(t, x))$, and not only its positivity, is known.

Theorem 4. Let $\Phi(x)$ be a directing function for system (2), having nonzero index (for example, let $\Phi(x)$ be even). Suppose

$$(\operatorname{grad} \Phi(x), f(t, x)) \geq a[\Psi(x)] \qquad (x \in T), \tag{7}$$

$$(\operatorname{grad} \Psi(x), f(t, x)) \geq -L[\Psi(x)] \qquad (x \in T), \tag{8}$$

where the functions $a(u)$, $L(u)$ are positive, the function $a(u)$ is nonincreasing for negative u and nondecreasing for positive u, and the continuously differentiable function $\Psi(x)$ satisfies the condition

$$\lim_{x\in T;\ \|x\|\to\infty} |\Psi(x)| = \infty. \tag{9}$$

Finally, suppose that for every positive a

$$\int_{-\infty}^{-a} \frac{a(u)\,du}{L(u)} = \int_{a}^{\infty} \frac{a(u)\,du}{L(u)} = \infty. \tag{10}$$

Then system (2) has at least one solution defined on the whole axis and uniformly bounded.

Theorem 5. Suppose that the right-hand sides of system (2) are ω-periodic in t and satisfy the conditions of theorem 4. Then the system has at least one ω-periodic solution.

4. From theorems 1, 3 and 5 one can obtain various criteria for the existence of periodic solutions for systems with variable delay. We shall use below the notation from (2).

For simplicity, consider a system of the form

$$\dot{x} = f\{t, x(t), x[t - h(t)]\}. \tag{11}$$

We shall assume that $f(t, x, y)$ and $h(t)$ have the property of ω-periodicity in t. To prove the existence of periodic solutions of system (11), consider the auxiliary family of equations

$$\dot{x} = f\{t, x(t), x[t - \lambda h(t)]\}, \tag{12}$$

where $\lambda \in [0, 1]$.

Lemma 1. Suppose that the right-hand side of system (11) satisfies the condition

$$(\operatorname{grad} \Phi(x), f(t, x, y)) > 0 \qquad (\|x\| \geq \rho_0), \tag{13}$$

$$(\operatorname{grad} \Psi_i(x), f(t, x, y)) \geq 0 \qquad (x \in T,\ i = 1, \dots, r), \tag{14}$$

where the functions $\Psi_1(x), \Psi_2(x), \dots, \Psi_r(x)$ satisfy condition (6).

Then the ω-periodic solutions of all systems (12) (if such solutions exist) are uniformly bounded.

Lemma 2. Suppose condition (13) is satisfied. Suppose the inequalities

$$(\operatorname{grad}\Phi(x), f(t,x,y)) \geq a[\Psi(x)] \qquad (x \in T), \tag{15}$$

$$(\operatorname{grad}\Psi(x), f(t,x,y)) \geq -L[\Psi(x)] \qquad (x \in T), \tag{16}$$

are satisfied, where the functions $a(u)$ and $L(u)$ satisfy the conditions indicated in Theorem 4. Then the ω-periodic solutions of all systems (12) (if such solutions exist) are uniformly bounded.

The periodic solutions of system (12) coincide (see (^{2})) with the solutions of various integro-functional equations with operators acting in the space C of functions continuous on $[0,\omega]$ (or in some subspace of it). Let us write down two such equations:

$$x = x(\omega) + \int_0^t f\{s, x(s), \tilde{x}_\omega[s - h(s)]\}\, ds, \tag{17}$$

$$x = e^t(1-e^\omega)^{-1}\int_0^\omega \{x(s) + f[s, x(s), \tilde{x}_\omega(s-h(s))]\}\, ds +$$

$$+ \int_0^t e^{t-s}\{x(s) + f[s, x(s), \tilde{x}(s-h(s))]\}\, ds, \tag{18}$$

where $\tilde{x}_\omega$ is the ω-periodic extension of the function $x(t) \in C$. The operator defined by the right-hand side of equation (18) is conveniently considered in the subspace $C_\omega \subset C$ of functions taking equal values at the endpoints of the interval $[0,\omega]$.

Under the conditions of Lemmas 1 and 2, the fixed points of the operators $Q(x;\lambda)$, defined by the right-hand side of one of equations (17) or (18), are uniformly bounded. Therefore, in order to apply the alternative principle of existence of periodic solutions (^{2}), it is necessary to show that the operator $Q(x;\lambda)$ is continuous jointly in the variables and compact.

Theorem 6. Suppose the conditions of Lemma 1 or Lemma 2 are satisfied. Suppose the index of the function $\Phi(x)$ is different from zero (for example, $\Phi(x)$ is even). Then system (11) has at least one ω-periodic solution.

5. It is interesting to note that the operator $Q(x;\lambda)$, defined by the right-hand side of equation (18), is continuous jointly in the variables under weaker restrictions than the operator defined by the right-hand side of equation (17). One can construct an arbitrarily smooth (having a prescribed number of derivatives) function $h(t)$ such that the right-hand side of equation (17) is not continuous in λ in C. The right-hand side of equation (18)

is continuous jointly in the variables already in the case when $h(t)$ is continuous.

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Note: Figure translations are in progress. See original paper for figures.

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