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Abstract

Full Text

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On One-to-One Mappings onto Metric Spaces

(Presented by Academician P. S. Aleksandrov on 24 IV 1964)

The general question answered by the theorems of this paper is the following: when, under a condensation,* is the preimage of a metric space metrizable?

Theorem 1. *Let a condensation $f : X \rightarrow Y$ be given, where X is a locally connected and peripherally bicomact **space, and Y is a metric space. Then X is metrizable.***

The space Y , by Bing's metrization criterion⁽¹⁾, has a σ -discrete base $\gamma = \{\gamma_i\}$, where each γ_i is a discrete system of open sets:

$$\gamma_i = \{U_{i\alpha}\}, \quad U_{i\alpha} \cap U_{i\beta} = \Lambda, \quad \alpha \neq \beta; \quad i = 1, 2, \dots$$

Denote by $V_{i\alpha}$ the $1/i$ -neighborhood of the set $U_{i\alpha}$, i.e. the set of all points whose distance from $U_{i\alpha}$ is not greater than $1/i$. The collection of all $V_{i\alpha}$ forms a base of the space Y ; denote it by $\omega = \{V_{i\alpha}\}$.

Let us note that every open subset of a locally connected space decomposes into open components. Consequently,

$$f^{-1}V_{i\alpha} = \{W_{i\alpha\delta}\},$$

where each $W_{i\alpha\delta}$ is open and connected,

$$W_{i\alpha\delta_1} \cap W_{i\alpha\delta_2} = \Lambda, \quad \delta_1 \neq \delta_2.$$

The union of all $W_{i\alpha\delta}$ over all indices i, α, δ is a system of open subsets of the space X ; denote it by W . We shall verify that W is a base in X .

Take an arbitrary point x with its neighborhood Ox . We must show that there is a $W_{i\alpha\delta}$ such that

$$x \in W_{i\alpha\delta} \subset Ox.$$

Take a connected neighborhood $O'x \subset Ox$ with bicomact boundary. Denote

$$F = [O'x] \setminus O'x.$$

The set F is bicomact, and therefore fF is also bicomact, not containing the point

$$y = fx.$$

There is an element $V_{i\alpha} \in \omega$ such that

$$y \in V_{i\alpha}, \quad V_{i\alpha} \cap fF = \Lambda.$$

Take that component of the set $f^{-1}V_{i\alpha}$ which contains x ; let it be $W_{i\alpha\delta_0}$. It is asserted that

$$W_{i\alpha\delta_0} \subset O'x.$$

If there were a point

$$a \in W_{i\alpha\delta_0} \setminus O'x,$$

then the set $W_{i\alpha\delta_0}$, being connected, would intersect F , but then

$$V_{i\alpha} \cap fF \neq \Lambda,$$

which is not the case. Thus,

$$W = \{W_{i\alpha\delta}\}$$

is a base of the space X .

Denote

$$Z_{i\alpha\delta} = W_{i\alpha\delta} \cap f^{-1}U_{i\alpha}.$$

It is asserted that

$$Z = \{Z_{i\alpha\delta}\}$$

is a base in X . Take an arbitrary point x with its neighborhood Ox . We must show that there is a $Z_{i\alpha\delta}$ such that

$$x \in Z_{i\alpha\delta} \subset Ox.$$

Take a connected neighborhood $O'x \subset Ox$ with bicomact boundary. Denote

$$F = [O'x] \setminus O'x.$$

The set F is bicomact, and therefore fF is also bicomact. There is an element $U_{i\alpha} \in \gamma$ such that

$$y \in U_{i\alpha} \quad \text{and} \quad V_{i\alpha} \cap fF = \Lambda,$$

where

$$y = fx.$$

Take that component of the set $f^{-1}V_{i\alpha}$ which contains x ; let it be $W_{i\alpha\delta_0}$. It is easy to verify that

$$W_{i\alpha\delta_0} \subset O'x,$$

and consequently

$$Z_{i\alpha\delta_0} \subset O'x,$$

and, since

$$x \in Z_{i\alpha\delta_0},$$

it has been proved that

$$Z = \{Z_{i\alpha\delta}\}$$

is a base of the space X .

It remains to prove that

$$Z = \{Z_{i\alpha\delta}\}$$

is a σ -discrete system of sets of the space X . Fix i and consider

$$Z_i = \{Z_{i\alpha\delta}\};$$

we shall prove that this is a discrete system of sets in X . First, this system

* A condensation is a one-to-one continuous mapping.

** A space is peripherally bicomact if it has a base whose elements have bicomact boundaries.

*** Theorem 1 overlaps with Mardesic' s theorem that a locally connected bicomactum mapped zero-dimensionally onto a compactum is metrizable.

disjoint. Suppose that it is not discrete, i.e., suppose that there is a point $x \in X$ such that every neighborhood Ox of it intersects more than one set of the system $Z_i = \{Z_{i\alpha\delta}\}$. Then, obviously, there is an α such that $y \in [U_{i\alpha}]$, where $y = fx$. Take an arbitrary neighborhood Oy of the point y , but such that $Oy \subset V_{i\alpha}$, and such that Oy intersects only one element of the system γ_i , namely $U_{i\alpha}$.

Now take that component of the set $f^{-1}Oy$ which contains the point x , and denote it by Ox . There is a δ_0 such that $Ox \subset W_{i\alpha\delta_0}$, and, therefore, the neighborhood Ox intersects only one element of the system $Z_i = \{Z_{i\alpha\delta}\}$, namely $Z_{i\alpha\delta_0}$. This contradicts the original assumption. Thus, $Z = \{Z_{i\alpha\delta}\}$ is a σ -discrete base in X , and, by Bing' s criterion, the space X is metrizable.

Theorem 2. *Suppose a condensation $f : X \rightarrow Y$ is given, where X is a locally connected, connected, and locally bicomact space, and Y is a Hausdorff, locally bicomact, locally connected, uncohesive space with a countable base. Then f is a homeomorphism.*

Corollary. *Suppose a condensation $f : X \rightarrow E^n$ is given, where X is a locally connected, connected, and locally bicomact space, and E^n is Euclidean space. Then f is a homeomorphism.*

Theorem 2 follows from considering Theorem 1 together with the main theorem of the paper (2).

A. V. Arkhangel' skii constructed an example of a condensation of a locally bicomact nonmetrizable space onto a metric space. This example shows that in Theorem 1 the condition of local connectedness of the space X is essential.

Theorem 3. *Suppose there is a condensation $f : X \rightarrow Y$, where X is a countably compact* space and Y is a metric space. Then f is a homeomorphism, and X is a compactum**.*

Proof. The continuous image of a countably compact space is countably compact. Consequently, Y is a compactum. It remains only to prove that f is a

homeomorphism. If a condensation is a closed mapping, then it is a homeomorphism. Suppose that f is not closed, i.e., that there is a closed set $A \subset X$ such that $B = fA$ is not closed in Y . In the compactum Y there is a countable subset $C \subset B$ whose unique limit point, the point a , lies outside B . Thus, $C = \{c_i\}$; for each point c_i choose a neighborhood Oc_i such that $[C] \cap Oc_i = c_i$.

Now denote $U = Y \setminus [C]$. Next, take a neighborhood Ob of the point $b = f^{-1}a$ such that $Ob \cap f^{-1}C = \Lambda$.

I assert that from the countable covering of the space X , consisting of $f^{-1}U$, Ob , and all $f^{-1}Oc_i$, $i = 1, 2, \dots$, it is impossible to choose a finite covering; this contradicts the countable compactness of the space X . The theorem is proved.

It follows from Theorem 3, for example, that the space of ordinal numbers up to ω_1 exclusively cannot be condensed onto a metric space.

In connection with Theorem 2 I pose a question, the answer to which is unknown to me. Is it possible to condense onto E^n a connected locally bicomact space of uncountable weight?

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REFERENCES

1. R. H. Bing, *Canad. J. Math.*, **3**, No. 2, 175 (1951).
2. V. Proizvolov, *DAN*, **151**, No. 6, 1286 (1963).

* A space is countably compact if from every countable covering of it one can choose a finite subcovering.

** *Note added in proof.* As has become known to me, a similar result, as well as some other results in this direction, were independently obtained by V. Klyushin.

Note: Figure translations are in progress. See original paper for figures.

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