ON THE STATISTICAL THEORY OF TURBULENCE

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Abstract

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HYDROMECHANICS

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ON THE STATISTICAL THEORY OF TURBULENCE

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1. In recent years, in the statistical theory of turbulence, as in a number of other branches of theoretical physics, methods analogous to the methods of quantum field theory have been used more and more. It was precisely by means of such methods that, in Wyld' s work (^1), a theory of turbulence of an incompressible fluid was formulated, and it was shown that, by selectively summing Feynman diagrams of a certain structure arising in the corresponding perturbation theory, the problem reduces to the solution of three exact integral equations for three functions, one of which is the spectral function of the velocity field. Each of the three equations mentioned has the form of an infinite series of integrals of powers of the unknown functions, and the approximations consist in a suitable truncation of these series so as to obtain integral equations with a finite number of terms.

In particular, Wyld showed that one of the simplest approximations, in which the simplest nontrivial terms of the series are retained only for two of the three functions mentioned above, leads to Kraichnan's statistical equations, describing stationary, homogeneous, and isotropic turbulence at large Reynolds numbers (^2). It is known, however, that Kraichnan' s equations give, for the inertial range, a spectrum different from the Kolmogorov spectrum.

In the present work it is shown that a somewhat more complicated approximation of Wyld' s equations leads to a system of equations that makes it possible to obtain the Kolmogorov spectrum.

2. If in Wyld' s equations the simplest nontrivial terms are retained not for two, but for all three functions, then a system of equations is obtained that may be represented in the following symbolic form:

$$\sim\!\sim\!\sim\!\sim \qquad - F_{ij}(k) - +2 \qquad - \bigcirc - \tag{1}$$

$$\bullet = \bullet + 4\,\triangle + 4\,\triangle + 4\,\triangle \tag{2}$$

$$\odot = \circ + 4\,\triangle \tag{3}$$

Here the bold wavy line corresponds to the spectral function $U_{ij}(k)$ of the velocity field; the bold straight line to the complete Green function $S'_{ij}(k)$; the bold dot to the first vertex function $\Gamma_{ijm}(k,k';k-k')$; the double circle to the second vertex function $\Gamma_{i\mu}(k,k')$; the simple dot to the first simple vertex $-\frac{i}{2}P_{ijm}(k)$, where $P_{ijm}(k) = k_m P_{ij}(k) + k_j P_{im}(k)$, and

$$P_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2};$$

the simple circle corresponds to the second simple vertex $\gamma_\mu(k,k')$, where $\gamma_\mu(k,k') = \{\nu(k+k'),-i\}$; finally, $F_{ij}(k)$–

the known spectral function of the components of the external forcing force, $k = \{\mathbf{k},\omega\}$.

To equations (1)–(3) one must also add the Ward integral identity

$$[S'_{ij}(k)]^{-1} - [S'_{ij}(k')]^{-1} = (k_\mu - k'_\mu)\Gamma'_{ij\mu}(k,k'). \tag{4}$$

It is not difficult to show that, within the approximation under consideration, equation (3) and the Ward relation (4) are equivalent to one equation for the complete Green function S'_{ij}, so that the system (1)–(4) takes the form

[[diagram: wavy line]] = [[diagram: straight line]] $F_{ij}(k)$ [[diagram: straight line]]+2 [[diagram: straight line wi

(5)

[[diagram: vertex dot]] = [[diagram: vertex dot]]+4 [[diagram: triangular vertex diagram]]

(6)

[[diagram: straight line]] = [[diagram: straight line]]+4 [[diagram: straight line with wavy-line loop insertion]]

(7)

where the thin straight line corresponds to the unperturbed Green function. Here, in addition, the second and third "triangles" from equation (2) have been omitted, since their contribution, as can be shown, proves to be inessential. Using the correspondence rules, one could write down the system of equations for the tensor functions S'_{ij}, U_{ij}, and Γ_{ijm} also in analytic form. However, in the case of stationary, homogeneous, and isotropic turbulence, such a system can be written for the corresponding scalar functions:

$$U(k) = |S'(k)|^2 \left[F(k) + \frac{k^2}{(2\pi)^4} \int d^4k_1 \cdot a(\mathbf{k}, \mathbf{k}_1, \mathbf{k} - \mathbf{k}_1) \times \right.$$

$$\left. \times\, U(k - k_1) U(k_1) \left| \Gamma(k, k_1; k - k_1) \right|^2 \right]; \tag{8}$$

$$\Gamma(k, k'; k - k') = 1 - \frac{k^2}{(2\pi)^4} \int d^4k_1 \cdot c(\mathbf{k}, \mathbf{k}', \mathbf{k} - \mathbf{k}'; \mathbf{k}_1) \Gamma(k, k - k_1; k_1) \times$$

$$\times\, S'(k - k_1) \Gamma(k - k_1, k'; k - k_1 - k') S'(k - k' - k_1) \times$$

$$\times\, \Gamma(k - k' - k_1, k - k', -k_1) U(k_1); \tag{9}$$

$$S'(k) = S(k) - S(k) \left[\frac{k^2}{(2\pi)^4} \int d^4k_1 \cdot b(\mathbf{k}, \mathbf{k}_1, \mathbf{k} - \mathbf{k}_1) \Gamma(k, k_1; k - k_1) \times \right.$$

$$\left. \times\, S'(k - k_1) U(k_1) \Gamma(k - k_1, -k_1; k) \right] S'(k). \tag{10}$$

In obtaining system (1) we set

$$U_{ij}(k) = U(k) P_{ij}(\mathbf{k}), \qquad F_{ij}(k) = F(k) P_{ij}(\mathbf{k}),$$

$$S'_{ij}(k) = S'(k) P_{ij}(\mathbf{k}), \qquad \Gamma_{ijm}(k, k'; k - k') = -\frac{i}{2} P_{ijm}(\mathbf{k}) \Gamma(k, k'; k - k'),$$

$S(k) \equiv (-i\omega + \nu k^2)^{-1}$ is the unperturbed Green function. The geometric factors in system (1) are respectively equal to

$$a(\mathbf{k}, \mathbf{k}_1, \mathbf{k} - \mathbf{k}_1) = \frac{1}{4k^2} P_{il\lambda}(\mathbf{k}) P_{\lambda\mu}(\mathbf{k} - \mathbf{k}_1) P_{l\rho}(\mathbf{k}) P_{i\rho\mu}(\mathbf{k}); \tag{11}$$

$$b(\mathbf{k}, \mathbf{k}_1, \mathbf{k} - \mathbf{k}_1) = \frac{1}{2k^2} P_{il\lambda}(\mathbf{k}) P_{\lambda\rho i}(\mathbf{k} - \mathbf{k}_1) P_{l\rho}(\mathbf{k}_1); \tag{12}$$

$$c(\mathbf{k}, \mathbf{k}', \mathbf{k} - \mathbf{k}'; \mathbf{k}_1) = \frac{k_i}{2k^4} P_{il\lambda}(\mathbf{k}) P_{lj\rho}(\mathbf{k} - \mathbf{k}_1) P_{\rho i\nu}(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}') P_{\lambda\mu}(\mathbf{k}_1). \tag{13}$$

Let us note that system (1) is analogous to the closed system of equations of quantum electrodynamics in the zeroth approximation in the seed charge (3).

3. If in system (1) one sets $\Gamma(k, k'; k-k') \equiv 1$, then equations (8) and (10) pass into Kraichnan' s equations for $U(k)$ and $S'(k)$*, and the principal contribution

in the integral terms, for example in equation (10), will indeed give, as Kraichnan believed, the region of large-scale pulsations. In particular, for large k we obtain for the Green' s function $S'(k)$ the following asymptotic equation ($b \simeq 1$):

$$S'(k) = S(k) - S(k)[S'(k)]^2 \frac{k^2}{(2\pi)^4} \int U(k_1)\, d^4k_1,$$

where the integral is taken over the energy-containing region, so that

$$\int U(k_1)\, d^4k_1 \simeq \frac{3}{2} v_0^2,$$

where v_0 is the velocity of the large-scale pulsations. It is precisely this parameter v_0 that enters into all of Kraichnan' s asymptotic expressions, so that, for example, the energy spectrum in the inertial range has a form different from the Kolmogorov one,

$$E(k) = f(0)(\varepsilon v_0)^{1/2} k^{-3/2},$$

where $f(0)$ is a universal constant.

4. However, in the case considered by us, owing to the presence of the factors Γ in (8) and (10), the contribution of the large-scale pulsations at large k turns out to be suppressed. Let us show this.

Thus, let us turn to equation (9) and verify that it admits a solution of the form $\Gamma(k, k'; k-k') \simeq 1$ for $k' \sim k$ and $\Gamma(k, k'; k-k') \to 0$ for $k' \ll k,\ k \to \infty$. Indeed, for small k the contribution of the integral term to $\Gamma(k, k'; k-k')$ is small because of the presence in it of the factor k^2. For large k (as before, $k' \sim k$) the geometrical factor (13) tends to zero in a substantial region of small k_1.

Let us now consider the case $k' \ll k,\ k \to \infty$. In this case one may put

$$\Gamma(k, k-k_1;\, k_1) \simeq \Gamma(k, k;\, 0) \simeq 1;$$

$$\Gamma(k-k_1-k,\ k-k',\ -k_1) \simeq \Gamma(k, k;\, 0) \simeq 1;$$

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* We note that Kraichnan' s equations can also be obtained by another route (see, for example, ([4],[5])).

$$\Gamma(k-k_1,\,k';\,k-k_1-k') \simeq \Gamma(k,k';\,k-k'); \quad S'(k-k_1) \simeq S'(k);$$

$$S'(k-k_1-k') \simeq S'(k), \qquad c(k,k',k-k',k_1) \simeq c(k,0,k;0) = 1.$$

As a result we obtain

$$\Gamma(k,k';\,k-k') = 1 - \Gamma(k,k';\,k-k')\frac{k^2}{(2\pi)^4}[S'(k)]^2 \int U(k_1)\,d^4k_1,$$

or

$$\Gamma(k,k';\,k-k') = \frac{1}{1+\dfrac{k^2}{(2\pi)^4}[S'(k)]^2\int U(k_1)\,d^4k_1}.$$

The second term in the denominator may be estimated as

$$\frac{k^2}{(2\pi)^4}[S'(k)]^2 U(k_1)\,d^4k_1 \sim k^2\frac{1}{\omega^2}v_0^2 \sim \frac{v_0^2}{v^2} \to \infty,$$

since $S'(k) \sim 1/\omega$ as $\omega \to \infty$, and $\omega \sim vk$, where v is the mean velocity of the small-scale pulsations; whence $\Gamma(k,k';\,k-k') \sim v^2/v_0^2 \to 0$ as $k \to \infty$. Thus, we see that the vertex function $\Gamma(k,k';\,k-k')$ proves to be a cutoff factor for the large-scale pulsations.

The results obtained indicate that the parameter v_0, characterizing the contribution of the large-scale pulsations, must drop out of the asymptotic expressions, and it should be replaced, for example, by the quantity $v = [kE(k)]^{1/2}$, the mean velocity of the small-scale pulsations. Making this replacement in the expression $E(k) = f(0)(\varepsilon v_0)^{1/2}k^{-3/2}$, we arrive at the Kolmogorov spectrum $E(k) \sim \varepsilon^{2/3}k^{-5/3}$.

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Note: Figure translations are in progress. See original paper for figures.