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Abstract

Full Text

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ON THE ACCURACY OF CALCULATING THE MOMENT OF INERTIA OF NUCLEI BY THE METHOD OF FORCED ROTATION

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This paper gives an estimate of the accuracy of determining the moment of inertia of a nucleus by means of the model of forced rotation. The estimate is based on the use of the method of approximate projection of functions, developed in [1].

The moment of inertia is the coefficient of $I(I+1)$ in the expansion of the energy of the system in powers of the angular momentum I:

$$E_I = E_0 + I(I+1)/2J + \cdots \tag{1}$$

As is well known (see [2]), the Inglis formula

$$J = 2\sum_i \frac{|\langle i|\hat{I}_x|0\rangle|^2}{E_i - E_0} \tag{2}$$

for the moment of inertia exactly describes the term quadratic in I in the expression for the energy of a rotating system characterized by states of internal motion $|i\rangle$ and the corresponding energies E_i. However, the very separation of the rotational and internal motions of a closed system is an approximation*. Additional errors arise because the form of the functions $|i\rangle$ and the values E_i are chosen more or less arbitrarily. On the other hand, determining the moment of inertia by means of formula (1) is not connected with the approximations mentioned above, and therefore the question of the accuracy of the forced-rotation model is meaningful.

Let $\hat{H}$ be the Hamiltonian of the system and $\hat{I}_x$ one of the projections of the angular-momentum operator, having in the second-quantization representation the form

$$\hat{H} = \sum_i \varepsilon_i a_i^+ a_i + \frac{1}{2}\sum_{ijkl}\langle ij|G|lk\rangle a_i^+ a_j^+ a_k a_l, \qquad \hat{I}_x = \sum_{ij}\langle i|j_x|j\rangle a_i^+ a_j. \tag{3}$$

Taking into account the increase of the energy with I, one may assert that the problem

$$\delta\langle\Psi, \hat{H}\Psi\rangle = 0, \qquad \langle\Psi, \Psi\rangle = 1, \qquad \hat{I}_x\Psi = I\Psi, \tag{4}$$

with no restrictions on the form of the functions Ψ, exactly determines the lowest of the states of the system with quantum numbers I, $I(I+1)$, corresponding to the operators $\hat{I}_x$ and $\hat{I}^2 = \hat{I}_x^2 + I_y^2 + I_z^2$.

In the model of forced rotation, instead of problem (4) one solves the problem

$$\delta\langle\Psi, \hat{H}\Psi\rangle = 0, \qquad \langle\Psi, \Psi\rangle = 1, \qquad \langle\Psi, \hat{I}_x\Psi\rangle = I, \tag{5}$$

in which Ψ are taken from some restricted class of functions (see [3]). These functions are not, generally speaking, eigenfunctions of the ope-

* The fact that the states $|i\rangle$ and energies E_i describe the internal motion of the system and not the motion of the system as a whole is most easily seen from the fact that the energy operator of a closed system $\hat{H}$ commutes with the operator $\hat{I}_x$, and consequently the operator $\hat{I}_x$ can connect only states with one and the same energy.

operators $\hat{I}^2$, $\hat{I}_x$. The solution of problem (5) can be improved if, instead of the functions Ψ, one takes the trial functions

$$\Phi_I = \frac{1}{2\pi i}\oint \frac{dz}{z}\, z^{(\hat{I}_x - I)\frac{1}{\hbar}}\, \Psi, \tag{6}$$

which are projections of the functions Ψ onto the space of eigenfunctions of the operator $\hat{I}_x$ with eigenvalue I (see, for example, (4)). The use of projected functions does not guarantee the elimination of all errors in the solution of problem (5), but it makes it possible to estimate the accuracy of its solution using the functions Ψ.

We shall assume, as in work (1), that the mean values over the functions Φ can be calculated by the saddle-point method. Then, in accordance with the results of work (1), one may assert that the correction to the solution of problem (5), associated with the use of projected functions, is determined by an additional term in the energy of the system

$$\Delta E = -\frac{1}{2\overline{\Delta\hat{I}_x^2}}\,[H\Delta\hat{I}_x^2 - \bar{H}\Delta\hat{I}^2]. \tag{7}$$

In formula (7) the averaging is performed over the function Ψ, which is the solution of the variational problem (5), and the operator $\Delta\hat{I}_x^2 = (\hat{I}_x - I)^2$. Formula

(7) is valid if $\overline{\Delta\hat{I}_x^2}/\hbar^2 \gg 1$. Let us recall that projection operators were used for the analysis of rotational motions of nuclei in papers (5, 6); however, the form of the projection operator chosen by the authors of these papers did not make it possible to solve the variational problem (4) in the class of projected functions. In fact, in papers (5, 6) the internal inconsistency of the deformed-nucleus model was shown for sufficiently large values of the deformation parameter (and, consequently, of the quantity $\overline{\Delta\hat{I}_x^2}/\hbar^2$).

Let problem (5) be solved by the method of a canonical transformation. In this case the function Ψ is the vacuum with respect to the operators α, related to the operators a by the formula

$$a_\nu = \sum_{\nu'}(u_{\nu\nu'}\alpha_{\nu'} + v_{\nu\nu'}\alpha^+_{\tilde{\nu}'}). \tag{8}$$

(The state $\tilde{\nu}$ is time-conjugate to the state ν.) The mean values over the function Ψ are determined with the aid of the following matrix elements*:

$$\langle a_i^+ a_i\rangle = v_i^2 + (u_i^2 - v_i^2)\sum_j |f_{ij}|^2,$$

$$\langle a_i^+ a_{\tilde{i}}^+\rangle = \langle a_{\tilde{i}} a_i\rangle^* = u_i v_i - 2u_i v_i \sum_j |f_{ii}|^2; \tag{9}$$

$$\langle a_i^+ a_j\rangle = (u_i v_j - v_i u_j) f_{ij}, \qquad \langle a_j^+ a_{\tilde{i}}^+\rangle = \langle a_{\tilde{j}} a_i\rangle^* = (u_i v_j + v_i u_j) f^*_{ij}. \tag{10}$$

The coefficients f_{ij} satisfy the relations

$$f_{ij} = -f_{\tilde{j}\tilde{i}}, \qquad f_{ij} + f^*_{ji} = 0, \qquad f_{ii} = 0 \tag{11}$$

and are determined from the variational equations, which have the form

$$\begin{aligned}(E_i + E_j)f_{ij} - \sum_{i'j'}\langle \tilde{i}j|G|\tilde{j'}i'\rangle(u_i u_j + v_i v_j)(u_{i'} u_{j'} + v_{i'} v_{j'}) f_{i'j'} + \\ + \sum_{i',j'}(\langle i\tilde{i'}|G|\tilde{j}j'\rangle - \langle i\tilde{i'}|G|\tilde{j'}j\rangle)(u_i v_j - v_i u_j)(u_{i'} v_{j'} - v_{i'} u_{j'}) = \\ = \omega(u_i v_j - v_i u_j)\langle i|j_x|j\rangle,\end{aligned} \tag{12}$$

$$\sum_{ij}\langle i|j_x|j\rangle(u_i v_j - v_i u_j) f^*_{ij} = I. \tag{13}$$

* The notation is the same as in work (3).

Here

$$E_i = \sqrt{(\tilde{\varepsilon}_i - \lambda)^2 + \Delta_i^2}, \qquad \Delta_i = \sum_j \langle i\tilde{i}|G|j\tilde{j}\rangle u_j v_j, \tag{14}$$

and the quantities u_i, v_i are connected with the parameters λ, Δ_i, E_i by the usual equations of the u, v-transformation method.

The quantities in formula (7) have the form

$$\begin{aligned}\overline{\Delta\hat{I}_x^2} &= \sum_{\substack{\nu\nu'\\ \mu\mu'}} \langle\nu|j_x|\nu'\rangle\langle\mu|j_x|\mu'\rangle \left[\langle a_\nu^+ a_{\mu'}\rangle\langle a_{\nu'} a_\mu^+\rangle - \langle a_\nu a_{\mu'}\rangle\langle a_{\nu'}^+ a_\mu^+\rangle\right] = \\ &= \sum_{\nu,\mu} \left|\langle\nu|j_x|\mu\rangle\right|^2 \left[v_\nu^2 u_\mu^2 - (u_\nu v_\nu)(u_\mu v_\mu)\right],\end{aligned} \tag{15}$$

$$\begin{aligned}\Delta\hat{I}_x^2 - \overline{\Delta\hat{I}_x^2} = \sum_{\substack{\nu\nu'\\ \eta\eta'}} \sum_{\substack{\mu\mu'\\ \rho\rho'}} \langle\nu|j_x|\nu'\rangle\langle\mu|j_x|\mu'\rangle u_{\nu\eta}^* v_{\nu'\eta'} u_{\mu\rho} v_{\mu'\rho'}^* \alpha_\eta^+ \alpha_{\eta'}^+ \alpha_\rho \alpha_{\rho'} + \\ + \text{ Hermitian conjugate.}\end{aligned} \tag{16}$$

It follows from formula (16) that the corrections determined by formula (7) are connected with the interaction Hamiltonian, since only it contains terms that can compensate the 4 creation operators α^+ in formula (16). This is connected with the possibility of exact diagonalization of the Hamiltonian of noninteracting particles by the method of the canonical transformation (8). It should be remembered, however, that the Bardin function, constructed on the single-particle functions of an anisotropic potential, in principle cannot be an exact eigenfunction of the Hamiltonian of a closed system.

Rather laborious calculations make it possible to write the following expression for the quantity ΔE:

$$\Delta E = -\frac{1}{4\overline{\Delta\hat{I}_x^2}} \sum_{ijkl} \langle ij|G|lk\rangle\langle kl|\Delta\hat{I}_x^2 - \overline{\Delta\hat{I}_x^2}|ji\rangle + \text{complex conjugate}, \tag{17}$$

where

$$\langle kl|\Delta\hat{I}_x^2 - \overline{\Delta\hat{I}_x^2}|ji\rangle = 8\sum_{\nu\nu'}\langle\nu|j_x|\nu'\rangle\langle a_k a_\nu^+\rangle\langle a_l a_{\nu'}\rangle\sum_{\mu\mu'}\langle\mu|j_x|\mu'\rangle\langle a_\mu a_j\rangle^*\langle a_i^+ a_{\mu'}\rangle -$$
$$- 4\sum_{\nu\nu'}\sum_{\mu\mu'}\langle\nu|j_x|\nu'\rangle\langle\mu|j_x|\mu'\rangle\left[\langle a_k a_\nu^+\rangle\langle a_j^+ a_{\nu'}\rangle\langle a_l a_{\mu'}^+\rangle\langle a_i^+ a_\mu\rangle + \right.$$
$$\left. + \langle a_\nu a_j\rangle^*\langle a_k a_{\nu'}\rangle\langle a_\mu a_i\rangle^*\langle a_l a_{\mu'}\rangle\right]. \tag{18}$$

The determination of the corrections to the moment of inertia is easily carried out by separating out the part quadratic in I from expression (18) with the aid of formulas (10). The resulting expressions contain a large number of different terms. If the interaction Hamiltonian has the form

$$\langle ij|G|lk\rangle = -\frac{G}{2}\lambda(i)\lambda(l)\delta_{i\tilde{j}}\delta_{l\tilde{k}}, \tag{19}$$

where

$$\lambda(i) = -\lambda(\tilde{i}), \qquad |\lambda(i)| = 1, \tag{20}$$

then the magnitude of the correction to the moment of inertia can be estimated by means of only the first term on the right-hand side of formula (18)

$$\Delta E = \frac{I^2 G}{2\overline{\Delta\hat{I}_x^2}}\xi_+\xi_- \simeq \frac{1}{2}\omega^2 J_0 \frac{J_0 G}{\overline{\Delta I_x^2}}\xi_+\xi_-. \tag{21}$$

In formula (21) the quantities

$$\xi_\pm = \left\{\sum_{i\mu}\frac{|\langle i|j_x|\mu\rangle|^2}{E_i + E_\mu}(u_i v_\mu - v_i u_\mu)\lambda(i)\left[(1 \pm (u_i^2 - v_i^2))(u_i u_\mu + v_i v_\mu) + 2u_i v_i(u_i v_\mu - v_i u_\mu)\right]\right\} \times$$
$$\times \left\{\sum_{i\mu}\frac{|\langle i|j_x|\mu\rangle|^2}{E_i + E_\mu}(u_i v_\mu - v_i u_\mu)^2\right\}^{-1} \tag{22}$$

are, in order of magnitude, quantities of order unity; $\omega = I/J_0$ is the rotation frequency corresponding to the angular momentum I, and

$$J_0 = \sum_{i\mu}\frac{|\langle i|j_x|\mu\rangle|^2}{E_i + E_\mu}(u_i u_\mu - v_i u_\mu)^2 \tag{23}$$

is the value of the moment of inertia found without projection. From formula (21) it follows that the correction to the moment of inertia is equal to

$$|\Delta J/J_0| \simeq GJ_0/\overline{\Delta\hat{I}_x^2}. \tag{24}$$

The character of the expression on the right-hand side of formula (24) is easiest to see if one uses formula (2) for J_0 and takes into account that the average energy of the excited states E_{av} contributing to the moment of inertia is, in order of magnitude, equal to twice the gap Δ. But, on the other hand:

$$J_0 = 2\sum_i \frac{|\langle i|j_x|0\rangle|^2}{E_i - E_0} = \frac{2}{E_{\text{av}}}\sum_i |\langle i|j_x|0\rangle|^2 = \frac{2\overline{\Delta\hat{I}_x^2}}{E_{\text{av}}}. \tag{25}$$

Therefore formula (24) can be rewritten in the form

$$|\Delta J/J_0| \sim G/\Delta.$$

It is thus seen that the order of magnitude of the correction is 10%. Formula (25) also makes it possible to estimate, from known data on the magnitude of single-particle excitations and from the values of the moment of inertia, the value of the parameter $\overline{\Delta\hat{I}_x^2}/\hbar^2$, which is a measure of the accuracy of the cranking method. For nuclei in the rare-earth group this quantity is of order 20–30, which attests to the applicability of the method.

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Note: Figure translations are in progress. See original paper for figures.