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V. P. SHUNKOV

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Abstract

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V. P. SHUNKOV

ON GROUPS DECOMPOSABLE INTO A UNIFORM PRODUCT OF THEIR p-SUBGROUPS

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In group theory, representations of groups in the form of various kinds of products (direct, semidirect, free, etc.) of their subgroups play a major role. The content of the present article is connected with the representation of a group in the form of such a product of its subgroups in which the cyclic subgroups of any two distinct factors are pairwise permutable. The problem of studying such representations was first posed by S. N. Chernikov (who called them uniform), who suggested to the author that he develop, in particular, the question of groups decomposable into a uniform product of their Sylow p-subgroups.

1. The structure of groups decomposable into a uniform product of their Sylow p-subgroups is described by the following theorem.

Theorem 1. *A periodic group $\mathfrak{G}$ is decomposable into a uniform product of its Sylow p-subgroups if and only if it is a semidirect product of a Hall abelian subgroup $\mathfrak{H}$, all subgroups of which are invariant in $\mathfrak{G}$, and some S-group $\mathfrak{R}$.*

A periodic group is called an S**-group** if it is decomposable into the direct product of its Sylow p-subgroups.

Corollary. Let $\mathfrak{G}$ be a group decomposable into a uniform product of its Sylow p-subgroups, and let $\mathfrak{H}$ be its subgroup defined by the properties indicated in Theorem 1. If p is an arbitrary prime from the set of prime divisors of the orders of elements of the subgroup $\mathfrak{H}$, then for each element $G \in \mathfrak{G}$ there exists a natural number $k = k(G)$, less than p, such that for every p-element $H \in \mathfrak{H}$ the relation

$$G^{-1}HG = H^{kp^{n-1}},$$

holds, where p^n is the order of the element H.

Theorem 2. *Every subgroup of a group decomposable into a uniform product of its Sylow p-subgroups is also decomposable into a uniform product of its Sylow p-subgroups.*

2. **Definitions.** Let $\mathfrak{R}$ and $\mathfrak{H}$ be some normal divisors of the group $\mathfrak{G}$, and let A, B be some elements of $\mathfrak{G}$. Define the natural number λ as follows: 1) $\lambda = 1$ if the orders of the elements $A\mathfrak{R} \in \mathfrak{G}/\mathfrak{R}$ and $B\mathfrak{H} \in \mathfrak{G}/\mathfrak{H}$ are relatively prime or infinite; 2) λ is equal to the product of all prime divisors

of the greatest common divisor of the orders of the elements $A\mathfrak{N} \in \mathfrak{G}/\mathfrak{N}$ and $B\mathfrak{H} \in \mathfrak{G}/\mathfrak{H}$. If the order of one of the elements $A\mathfrak{N} \in \mathfrak{G}/\mathfrak{N}$ and $B\mathfrak{H} \in \mathfrak{G}/\mathfrak{H}$ is finite and the other is infinite, then we shall take the number λ to be equal to the product of all prime divisors of the order of that element $A\mathfrak{N} \in \mathfrak{G}/\mathfrak{N}$, $B\mathfrak{H} \in \mathfrak{G}/\mathfrak{H}$ whose order is finite.

We shall call the elements A and B of the group $\mathfrak{G}$ **comparable with respect to the pair** $(\mathfrak{N}, \mathfrak{H})$ (relative to the pair $(\mathfrak{N}, \mathfrak{H})$) if $\mathfrak{N}$ and $\mathfrak{H}$ are normal divisors of the group $\mathfrak{G}$ and, for every element $G \in \mathfrak{G}$, the relations

$$(G\mathfrak{N})^{-1}A\mathfrak{N}(G\mathfrak{N}) = (A\mathfrak{N})^k, \qquad (G\mathfrak{H})^{-1}B\mathfrak{H}(G\mathfrak{H}) = (B\mathfrak{H})^l, \qquad k \equiv l \pmod{\lambda},$$

hold, where λ is the number mentioned above.

We shall call a group $\mathfrak{G}$ a ZH-group if it has an invariant series with abelian factors

$$E = \mathfrak{Z}_0 \subset \mathfrak{Z}_1 \subset \cdots \subset \mathfrak{Z}_\alpha \subset \cdots \subset \mathfrak{Z}_\gamma = \mathfrak{G},$$

in which, for any elements A and B of $\mathfrak{G}$ distinct from the identity, there exist two subgroups $\mathfrak{Z}_\alpha, \mathfrak{Z}_\beta$ such that $A \in \mathfrak{Z}_\alpha$, $B \in \mathfrak{Z}_\beta$, and A, B are comparable with respect to the pair $(\mathfrak{Z}_\alpha, \mathfrak{Z}_\beta)$.

Obviously, a group possessing an ascending central series is a special case of a ZH-group.

Theorem 3. *A finite group is a ZH-group if and only if it decomposes into a uniform product of its Sylow p-subgroups.*

Theorem 3 is false for infinite ZH-groups. A counterexample can be extracted from the work (1).

3. Definition. An automorphism φ of a group $\mathfrak{G}$ will be called **uniform** if, for every element $G \in \mathfrak{G}$, the relation

$$\varphi(G) = G^{K(G,\varphi)}$$

holds, where $K(G, \varphi)$ is an integer depending on G and φ.

Theorem 4. *A periodic group $\mathfrak{G}$ decomposes into a uniform product of its p-subgroups (not necessarily Sylow p-subgroups) if and only if it is a semidirect product of two subgroups: an S-group $\mathfrak{B}$ and an abelian subgroup $\mathfrak{A}$, satisfying the following conditions:*

1) $\mathfrak{A}$ *decomposes into the direct product*

$$\mathfrak{A} = \prod_{\alpha \in \mathfrak{M}} \times \mathfrak{A}_\alpha \qquad (1)$$

of subgroups $\mathfrak{A}_\alpha$ invariant in $\mathfrak{G}$ (not necessarily one for each p), such that, if p is an arbitrarily given prime divisor of the orders of the elements of the subgroup $\mathfrak{A}$, then every element $G \in \mathfrak{G}$ whose order is not divisible by p induces a uniform automorphism in any factor of the decomposition (1) which is a p-subgroup for this given p;

2) *every factor* $\mathfrak{A}_\alpha$ *of the decomposition (1) decomposes into a product of cyclic subgroups invariant in* $\mathfrak{G}$.

The example constructed below shows that, in a group decomposable into a uniform product of p-subgroups, not every subgroup decomposes into a product of this kind.

Example. Let $\mathfrak{G}$ be the group generated by elements A, B, C satisfying the following defining relations:

$$A^{p^2} = B^p = C^{qp} = 1,$$

where p and q are primes and $q/p-1$, and the relations

$$AB = BA, \qquad C^{-1}AC = A^k, \qquad C^{-1}BC = A^pB^{k,}$$

where k is a primitive root of the equation

$$x^{pq} \equiv 1 \pmod{p^2}.$$

Denote by D the element $D = AB$. It is easy to check that the group $\mathfrak{G}$ decomposes into a uniform product of p-subgroups:

$$\mathfrak{G} = \{A\}\{D\}\{C^q\}\{C^p\}.$$

However, its subgroup

$$\mathfrak{G}_1 = \{A^p, B, C\}$$

does not decompose into a uniform product of its p-subgroups.

4. Definition. We shall call a group $\mathfrak{G}$ a **quasi-uniform product of its subgroups** $[\mathfrak{A}_\alpha]$, $\alpha \in \mathfrak{M}$, if: 1) the group $\mathfrak{G}$ is generated by the subgroups $[\mathfrak{A}_\alpha]$, $\alpha \in \mathfrak{M}$; 2) in each subgroup $\mathfrak{A}_\alpha$ there exists a system of generators $[A_{\alpha\gamma}]$, $\gamma \in \mathfrak{M}_\alpha$, such that the subgroups $\{A_{\alpha\gamma}\}$, $\{A_{\beta,\gamma'}\}$, where $\alpha \neq \beta,\ \gamma' \in \mathfrak{M}_\beta$, are permutable.

Theorem 5. A finite group is supersolvable if and only if it decomposes into a quasi-uniform product of its Sylow p-subgroups.

Corollary. A finite supersolvable group with abelian Sylow p-subgroups decomposes into a uniform product of cyclic p-subgroups.

5. Definition. Following Zappa (see ([2])), a finite group $\mathfrak{G}$ of order $o(\mathfrak{G})$ will be called a **MacLane group** if, for any subgroup $\mathfrak{H}$ of it and any divisor k of the number $o(\mathfrak{G})$ divisible by the order of the subgroup $\mathfrak{H}$, there exists in $\mathfrak{G}$ at least one subgroup of order k containing $\mathfrak{H}$.

Definition. Let $\mathfrak{G}$ be a nilpotent group and

$$E = \mathfrak{Z}_0 \subset \mathfrak{Z}_1 \subset \cdots \subset \mathfrak{Z}_{n-1} \subset \mathfrak{Z}_n = \mathfrak{G}$$

be its upper central series. We shall call the group $\mathfrak{G}$ **centrally dense** if, for every normal divisor $\mathfrak{N}$ of it, one can indicate an i ($i = 0, 1, \dots, n-1$) such that

$$\mathfrak{Z}_i \subseteq \mathfrak{N} \subseteq \mathfrak{Z}_{i+1}.$$

Obviously, in a finite centrally dense group every one of its Sylow p-subgroups is centrally dense.

Theorem 6. A finite MacLane group $\mathfrak{G}$ is a semidirect product of an invariant Hall centrally dense subgroup $\mathfrak{H}$ and some nilpotent subgroup $\mathfrak{N}$. The subgroup $\mathfrak{H}$ satisfies the following conditions:

1) if a Sylow p-subgroup $\mathfrak{P}$ of $\mathfrak{H}$ is abelian, then for each element $G \in \mathfrak{G}$ there exists a natural number $r = r(G)$, less than p, such that for any element $P \in \mathfrak{P}$ the relation

$$G^{-1}PG = P^{rp^{s-1}},$$

holds, where p^s is the order of the element P.

2) if a Sylow p-subgroup $\mathfrak{P}$ of $\mathfrak{H}$ is noncommutative and

$$E = \mathfrak{Z}_0 \subset \mathfrak{Z}_1 \subset \cdots \subset \mathfrak{Z}_{n-1} \subset \mathfrak{Z}_n = \mathfrak{P}$$

is its upper central series, then for each element $G \in \mathfrak{G}$ there exists a natural number $r = r(G)$, less than p, such that for any element

$$P\mathfrak{Z}_{i-1} \in \mathfrak{Z}_i/\mathfrak{Z}_{i-1} \quad (i = 1, \dots, n)$$

the relation

$$(G\mathfrak{Z}_{i-1})^{-1}P\mathfrak{Z}_{i-1}(G\mathfrak{Z}_{i-1}) = (P\mathfrak{Z}_{i-1})^{r^{n-i+1}}.$$

It is not difficult to verify that a finite group decomposable into a uniform product of its Sylow p-subgroups is a MacLane group.

Remark. In ([2]) the following assertion is given without proof. Let a MacLane group $\mathfrak{G}$ of order p^3q^β (p and q are distinct prime numbers) have a unique Sylow p-subgroup $\mathfrak{P}$. Then $\mathfrak{P}$ contains a normal divisor $\mathfrak{N}$ consisting of all elements of $\mathfrak{P}$ whose normalizer orders in $\mathfrak{G}$ are divisible by q^β.

The author of the present article has an example of a MacLane group of order $3^4 \cdot 2$ for which this assertion is false. If one excludes the trivial case in which $\mathfrak{P}$ is a direct factor in $\mathfrak{G}$, then from the theorem stated above it follows that the subgroup $\mathfrak{N}$ exists only when the length n of the upper central series of the subgroup $\mathfrak{P}$ does not exceed the order m of the group $\mathfrak{G}/\mathfrak{P}\mathfrak{Z}(\mathfrak{P})$, where $\mathfrak{Z}(\mathfrak{P})$ is the centralizer of the subgroup $\mathfrak{P}$; moreover, if $n = m$, then the subgroup $\mathfrak{N}$ coincides with the center of the subgroup $\mathfrak{P}$, and if $n < m$, with the identity subgroup.

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Ural State University
named after A. M. Gorky

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Note: Figure translations are in progress. See original paper for figures.