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Abstract

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ON UNIVERSAL BICOMPACTA OF A GIVEN WEIGHT AND A GIVEN DIMENSION

(Presented by Academician P. S. Aleksandrov on 8 X 1963)

P. S. Aleksandrov, in connection with E. Sklyarenko's theorem on the possibility of embedding a normal space of a given weight and a given dimension* in a bicompactum of the same weight and the same dimension ([1]), posed the question of the existence, among bicompacta of a given weight and a given dimension, of a universal one.**

At the opening of the Fourth All-Union Topological Conference on 24 IX 1963, Yu. M. Smirnov reported that A. Zarelua had proved the existence of a universal bicompactum of a given weight and a given dimension. In the present note an elementary proof of this fact is given. This proof was presented at the conference on 25 IX and is based on ideas entirely different from those of A. Zarelua's proof (which became known to me only on 28 IX from A. Zarelua' s report ([3])).

Theorem 1. *For every cardinal τ and every number $n = 0, 1, 2, \ldots$, there exists an n-dimensional bicompactum Π^n_τ of weight τ, containing a topological image of every n-dimensional normal space of weight τ. Moreover, the bicompactum Π^n_τ contains a topological image of every completely regular space X of weight τ whose maximal bicompact extension βX has dimension n.****

Proof. All completely regular spaces X of weight τ with $\dim \beta X = n$ split into classes $\alpha \in \mathfrak{A}$ of pairwise homeomorphic spaces. From each class α choose a space X_α and consider the space Y, which is the discrete sum of the maximal bicompact extensions βX_α of the spaces X_α, i.e. the bicompacta βX_α are pairwise disjoint and open-and-closed in Y. The space Y is evidently locally bicompact and normal (and even paracompact). Since $\dim \beta X_\alpha = n$ for every $\alpha \in \mathfrak{A}$, we also have $\dim Y = n$, i.e., since Y is a normal space, $\dim \beta Y = n$.

The weight of each space X_α is τ, i.e. there exists a homeomorphism f_α of each space X_α into the Tikhonov cube I^τ of weight τ. Since βX_α are maximal bicompact extensions, there exist extensions $\tilde{f}_\alpha : \beta X_\alpha \to I^\tau$ of the mappings f_α. Denote by f the mapping of the space Y into I^τ which coincides on each bicompactum βX_α with the mapping $\tilde{f}_\alpha$, and denote by $\tilde{f}$ the extension of the mapping f to βY. From the construction of the mapping f it follows that it will be a homeomorphism on each set $X_\alpha \subseteq \beta X_\alpha \subseteq Y \subseteq \beta Y$. By Theorem 2 of Mardešić from ([2]), there exists a bicompactum Π^n_τ of weight $\tau = w(I^\tau)$****

and dimension $n = \dim \beta Y$, and mappings

$$g : \beta Y \to \Pi_\tau^n \quad \text{and} \quad h : \Pi_\tau^n \to I^\tau,$$

such that $\tilde{f} = h \cdot g$. But then from the homeomorphism of the mapping $\tilde{f}$ on the sets X_α it follows that the mapping g is homeomorphic

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* By the dimension of a normal space X throughout this article is meant the dimension $\dim X$, defined by means of (locally) finite coverings.

** A space X is called **universal for a given class of spaces** μ if every space in the class μ has a homeomorphic mapping into X.

*** We note that for a normal space X one always has $\dim X = \dim \beta X$.

**** $w(X)$ denotes the weight of the space X.

on these sets. Thus, the bicompactum $\prod_\tau^n$ contains homeomorphic images of all the sets X_α. The theorem is proved.

Remark 1. A. V. Arhangel' skii and O. V. Lokutsievskii observed that the bicompactum $\prod_\tau^n$ may be regarded as dyadic, since, by a theorem of O. V. Lokutsievskii from (3), there exists a dyadic bicompactum $D^{n\tau}$ containing $\prod_\tau^n$, with

$$\dim D^{n\tau} = \dim \prod_\tau^n = n$$

and

$$w(D^{n\tau}) = w\left(\prod_\tau^n\right) = \tau.$$

Theorem 1 can be generalized somewhat with the aid of Theorems 2 and 10 from (4).

Theorem 2. *Among all bicompacta of weight χ and dimension n possessing zero-dimensional mappings onto bicompacta of weight τ, there exists a universal bicompactum $\prod_{\chi\tau}^n$. The bicompactum $\prod_{\chi\tau}^n$ will be a universal space for those and only those completely regular spaces which possess a resolving mapping of weight χ (4) into an n-dimensional bicompactum of weight τ.*

The method used in the proof of Theorem 1 is also suitable for proving the existence of universal metric spaces of a given weight and a given dimension; only, instead of Mardešić' s theorem, the following factorization theorem (5) is invoked:

Theorem 3. *Let there be a mapping f of a completely regular space X with $\dim \beta X = n$ onto a metric space S of weight τ. Then there exists an n-dimensional metric space R of weight τ and mappings $g : X \to R$ and $h : R \to S$ such that $f = h \cdot g$.*

From Theorem 3 it is quite easy to derive the theorem previously proved by Nagata in ([6]).

Theorem 4. *Among the n-dimensional metric spaces of weight τ there exists a universal one.*

Proof. As in Theorem 1, take a system of n-dimensional metric spaces X_α, $\alpha \in \mathfrak{A}$, of weight τ, such that the spaces X_α are pairwise nonhomeomorphic and every n-dimensional metric space of weight τ is homeomorphic to one of the spaces X_α. Denote by X the discrete sum of the spaces X_α. It is clear that the space X is metric and $\dim X = n$. Each space X_α has a homeomorphic mapping f_α into the generalized Hilbert space H^τ of weight τ (which is a metric space) ([7]). Denote by f the mapping of the space X into H^τ which coincides on each set X_α with f_α. It is clear that the mapping f is a homeomorphism on each set X_α. The rest (as in Theorem 1) follows from Theorem 3. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.