THEORY OF A CLASS OF DISCRETE SELF-CORRECTING SYSTEMS

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

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THEORY OF A CLASS OF DISCRETE SELF-CORRECTING SYSTEMS

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Consider a dynamic system described by the following system of differential equations

$$dx_i/dt = f_i\,[t, x_1, \dots, x_n, u_1(t), \dots, u_r(t), \xi_1, \dots, \xi_p], \tag{1}$$

$$0 \leq t \leq T \quad (i = 1, 2, \dots, n).$$

Here $x_1, \dots, x_n$ are the phase coordinates of the system; $u_1(t), \dots, u_r(t)$ are control functions; $\xi_1, \dots, \xi_p$ are certain constants. The boundary conditions and the duration of motion T may also depend on the constants $\xi_1, \dots, \xi_p$. In the realization of the motion, the values of the constants $\xi_1, \dots, \xi_p$ may differ from the nominal values, as a result of which the problem of correction arises. All the subsequent reasoning is based on the assumption that there exist time instants $t_1, \dots, t_m$ at which either the functions $u_1(t), \dots, u_r(t)$ have discontinuities of the first kind, or time intervals begin on which the variation of these functions is large. This assumption and the order of numbering of the time instants $t_1, \dots, t_m$ are illustrated in Fig. 1. It is assumed that the functions $u_1(t), \dots, u_r(t)$ admit a single-valued continuation when the time instants $t_1, \dots, t_m$ are varied.

Fig. 1

We shall monitor the value of the function

$$S = S[x_1(T), \dots, x_n(T), T]. \tag{2}$$

The quantity S, obviously, depends on the values of the constants $\xi_1, \dots, \xi_p$ and on the time instants $t_1, \dots, t_m$.

Fig. 2

Figure 2: Fig. 2

Let, for the values $\xi_1 = \bar{\xi}_1, \dots, \xi_p = \bar{\xi}_p$ and $t_1 = \bar{t}_1, \dots, t_m = \bar{t}_m$, the nominal regime be realized in the system, in which $S = \bar{S}$. Perturbed regimes are realized for $\xi_1 = \bar{\xi}_1 + \Delta\xi_1, \dots, \xi_p = \bar{\xi}_p + \Delta\xi_p$. In this case, if correction is not provided, then $S = \bar{S} + \Delta S$. We shall determine the conditions under which, by shifting the time instants $t_1, \dots, t_m$ by the quantities $\Delta t_1, \dots, \Delta t_m$, one can completely compensate the influence of the disturbances $\Delta\xi_1, \dots, \Delta\xi_p$ on the value of the function S.

We shall assume that in the course of the motion measurements are made of certain functions $n_1, \dots, n_q$, depending directly or indirectly on the constants $\xi_1, \dots, \xi_p$. At the same time, consider a function of these functions

$$N = N(t, n_1, \dots, n_q). \tag{3}$$

Denote the values of the function N in the nominal regime at $t = t_i$ by $\bar{N}_i$. In the presence of disturbances $\Delta\xi_1, \dots, \Delta\xi_p$, the time instants $t_1, \dots, t_m$

we shall determine from the equalities

$$N(t_i) = \lambda_i(t_i) \qquad (i = 1, 2, \dots, m), \tag{4}$$

where $\lambda_i(t)$ are functions to be determined. Obviously, $\lambda_i(\bar{t}_i) = \bar{N}_i$. If, proceeding from the condition $\Delta S = 0$, the functions $\lambda_i(t)$ can be determined independently of the disturbances $\Delta\xi_1, \dots, \Delta\xi_p$, then the system under consideration will be self-correcting with respect to the function S. Systems of this type are currently used in engineering (see, for example, (1)).

Fig. 2

Let us establish the conditions under which the equality $\Delta S = 0$ can be satisfied, and set forth a method for selecting the parameters of such systems for the case when the disturbances $\Delta\xi_1, \dots, \Delta\xi_p$ may be regarded as small. In this case one may put

$$\lambda_i(t) = \bar{N}_i + M_i(t - \bar{t}_i), \tag{5}$$

where M_i are constants to be determined. We shall denote the values of the function $N(t)$ at $t = t_i$ by N_i. The meaning of the quantities $\bar{N}_i, N_i, M_i$ and of the functions $\lambda_i(t)$ is explained in Fig. 2.

Taking into account that the quantities N_i, in addition to their dependence on $\Delta\xi_1, \dots, \Delta\xi_p$, also depend on $t_1, \dots, t_{i-1}, t_i$, the expressions for ΔN_i are written in the form

$$\Delta N_i = \frac{\partial N_i}{\partial t_1}\Delta t_1 + \dots + \frac{\partial N_i}{\partial t_{i-1}}\Delta t_{i-1} + \frac{\partial N_i}{\partial t_i}\Delta t_i + \sum_{k=1}^{p}\frac{\partial N_i}{\partial \xi_k}\Delta \xi_k. \tag{6}$$

Taking account of equalities (5) and (6), equations (4) may be written as

$$\frac{\partial N_i}{\partial t_1}\Delta t_1 + \dots + \frac{\partial N_i}{\partial t_{i-1}}\Delta t_{i-1} + \left(\frac{\partial N_i}{\partial t_i} - M_i\right)\Delta t_i = -\sum_{k=1}^{p}\frac{\partial N_i}{\partial \xi_k}\Delta \xi_k \tag{7}$$

$$(i = 1, 2, \dots, m).$$

All derivatives appearing here are calculated for the conditions of the nominal regime. With the aid of these equations, all increments $\Delta t_1, \dots, \Delta t_m$ can be expressed in terms of the disturbances $\Delta \xi_1, \dots, \Delta \xi_p$, provided only that

$$\partial N_i / \partial t_i \neq M_i \qquad (i = 1, 2, \dots, m). \tag{8}$$

In cases where this condition is not satisfied, equality (4) cannot be fulfilled on the linear section of the function $\lambda_i(t)$. Therefore, in what follows condition (8) will be assumed to hold. In virtue of equations (7) one may write

$$\Delta t_i = \sum_{k=1}^{p}\frac{\partial \Delta t_i}{\partial \xi_k}\Delta \xi_k \qquad (i = 1, 2, \dots, m), \tag{9}$$

where $\partial \Delta t_i / \partial \xi_k$ are determined, according to (7), from the equalities

$$\frac{\partial N_i}{\partial t_1}\frac{\partial \Delta t_1}{\partial \xi_k} + \dots + \frac{\partial N_i}{\partial t_{i-1}}\frac{\partial \Delta t_{i-1}}{\partial \xi_k} + \left(\frac{\partial N_i}{\partial t_i} - M_i\right)\frac{\partial \Delta t_i}{\partial \xi_k} = -\frac{\partial N_i}{\partial \xi_k} \tag{10}$$

$$(i = 1, 2, \dots, m).$$

Taking (9) into account, the expression for the increment of the function S is readily transformed to the form

$$\Delta S = \sum_{k=1}^{p}\left(\frac{\partial S}{\partial t_1}\frac{\partial \Delta t_1}{\partial \xi_k} + \dots + \frac{\partial S}{\partial t_m}\frac{\partial \Delta t_m}{\partial \xi_k} + \frac{\partial S}{\partial \xi_k}\right)\Delta \xi_k. \tag{11}$$

The condition that the perturbation $\Delta \xi_k$ have no influence on S is accordingly written in the form

$$\frac{\partial S}{\partial t_1}\frac{\partial \Delta t_1}{\partial \xi_k} + \dots + \frac{\partial S}{\partial t_m}\frac{\partial \Delta t_m}{\partial \xi_k} = -\frac{\partial S}{\partial \xi_k}. \tag{12}$$

Thus, for each fixed value of the index k, equations (10) and (12) constitute a linear system of $m+1$ equations for the m unknowns $\partial\Delta t_1/\partial\xi_k, \dots, \partial\Delta t_m/\partial\xi_k$. The determinant composed of the coefficients of the left-hand sides of equations (10) is equal to the product

$$\prod_{i=1}^{n}\left(\frac{\partial N_i}{\partial t_i} - M_i\right) \neq 0,$$

which, by virtue of condition (8), is different from zero.

As a consequence of this, the condition for the solvability of the system of equations (10) and (12) is the equality

$$\begin{vmatrix} \delta_1 & 0 & \cdots & 0 & \partial N_1/\partial\xi_k \\ \partial N_2/\partial t_1 & \delta_2 & \cdots & 0 & \partial N_2/\partial\xi_k \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \partial N_m/\partial t_1 & \partial N_m/\partial t_2 & \cdots & \delta_m & \partial N_m/\partial\xi_k \\ \partial S/\partial t_1 & \partial S/\partial t_2 & \cdots & \partial S/\partial t_m & \partial S/\partial\xi_k \end{vmatrix} = 0 \quad (k = 1, 2, \dots, p), \quad (13)$$

where

$$\delta_1 = \partial N_1/\partial t_1 - M_1, \dots, \delta_m = \partial N_m/\partial t_m - M_m.$$

This equality, when written for each of the perturbations $\Delta\xi_1, \dots, \Delta\xi_p$, gives a system of equations for determining the constants $M_1, \dots, M_m$ and the form of the dependence (3) of the function N on the measured functions $n_1, \dots, n_q$.

Let us transform the system of equations (13). First of all, note that multiplying any row of the determinant (13) by a certain constant is equivalent to multiplying the corresponding one of the equalities (4) by the same constant and, consequently, does not affect the determination of the time instants $t_1, \dots, t_m$. Therefore, when forming linear combinations of the rows of determinant (13), the values of the coefficients may be taken equal to one. Taking this into account and using the known theorem that a determinant is equal to zero if and only if there is a linear dependence among its rows, the equalities (13) may be replaced by the following system of equalities:

$$M_i = \frac{\partial}{\partial t_i}\left(S + \sum_{l=i}^{m} N_l\right) \quad (i = 1, 2, \dots, m), \quad (14)$$

$$\frac{\partial}{\partial\xi_k}\left(S + \sum_{i=1}^{m} N_i\right) = 0 \quad (k = 1, 2, \dots, p). \quad (15)$$

For a known function N, the solution of the problem of determining the constants $M_1, \dots, M_m$ is obviously given by formulas (14). To determine the dependence of the function N on the functions $n_1, \dots, n_q$, equations (15) serve. An important case is that of a linear dependence of the function N on the measured functions

$$N_i = \sum_{l=1}^{q} \beta_{il} n_l(t_i) \quad (i = 1, 2, \dots, m), \tag{16}$$

where β_{il} are constants to be determined. In this case the equalities (4), (5), which serve to determine the time instants $t_1, \dots, t_m$,

can be written in the form

$$\sum_{l=1}^{q} \beta_{il} \left[n_l(t_i) - \bar{n}_l(\bar{t}_i)\right] = M_i \left(t_i - \bar{t}_i\right) \quad (i = 1, 2, \dots, m), \tag{17}$$

where bars denote the nominal values of the quantities. Substitution of equalities (16) into equations (15) gives a system of p linear equations in the qm unknowns β_{il} $(i = 1, 2, \dots, m;\ l = 1, 2, \dots, q)$, whose coefficients are the derivatives $\partial n_l(t_i)/\partial \xi_k$, computed for the conditions of the nominal regime.

If $p \leqslant qm$, then the number of equations does not exceed the number of unknowns, and equations (15) can be satisfied with more or less arbitrary measured functions. It should be borne in mind, however, that there exist measured functions such that equations (15) can be satisfied also when $p > qm$. In particular, this occurs for

$$N_1 = \dots = N_{m-1} = 0; \qquad N_m = -S(t_1, \dots, t_m, \xi_1, \dots, \xi_p). \tag{18}$$

In this case equalities (14) give $M_1 = \dots = M_m = 0$. By virtue of (4) and (5), this permits the conclusion that, with such a choice of measured functions, the time instants $t_1, \dots, t_{m-1}$ may be determined arbitrarily, while the correction is effected by the subsequent shift of the time instant t_m in accordance with the equality

$$S(t_1, \dots, t_m, \xi_1, \dots, \xi_p) = \bar{S}.$$

A number of further analogous solutions of equations (15) can be indicated. It should be noted, however, that to determine functions N_i of the form (18), very complete information about the motion is required.

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CITED LITERATURE

1. Qian Xuesen, *Engineering Cybernetics*, IL, 1956.

Note: Figure translations are in progress. See original paper for figures.