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Abstract

Full Text

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**ON THE SOLVABILITY OF DIFFERENCE EQUATIONS
IN THE WHOLE SPACE**

(Presented by Academician I. G. Petrovskii on 15 X 1963)

In the present note we consider certain properties of solutions of linear difference equations with constant coefficients

$$Lu \equiv \sum_{\lambda=1}^{\Lambda} a_{\lambda} u(n_1 + k_{\lambda 1}, \dots, n_m + k_{\lambda m}) = 0; \quad (1)$$

$$Lu = \begin{cases} 1, & \text{if } n_1 = \dots = n_m = 0, \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

and, finally,

$$Lu = \psi(n_1, \dots, n_m), \quad (3)$$

where $u(n_1, \dots, n_m)$ is the unknown, and $\psi(n_1, \dots, n_m)$ is a given function of $m \geq 1$ variables taking arbitrary integer values; the numbers $k_{\lambda 1}, \dots, k_{\lambda m}$ ($1 \leq \lambda \leq \Lambda \geq 2$) are fixed integers; $a_{\lambda} \neq 0$ are arbitrary (complex) constants.

Using induction on m , it is easy to prove that equation (3) has at least one solution; equation (1) has no finite solutions different from the identically zero one; for $m \geq 2$, for any arbitrarily rapidly increasing continuous function $\varphi(t)$, there exists a solution of equation (3) not satisfying the estimate $u(n_1, \dots, n_m) \leq \varphi(n_1 + \dots + n_m)$ for arbitrarily large values of $n_1 + \dots + n_m$. (Here the coefficients a_{λ} may even depend on n_1, \dots, n_m .)

Here we shall indicate (Theorem 2) a priori assumptions on the behavior of the solution and of the right-hand side at infinity which entail unique solvability of equations (1)–(3) in the whole space, as well as of the Cauchy problem for these equations.

Consider the characteristic function

$$P(\bar{z}) = \sum_{\lambda=1}^{\Lambda} a_{\lambda} z_1^{k_{\lambda 1}} \dots z_m^{k_{\lambda m}}$$

(the bar above denotes a vector of m components) and the corresponding trigonometric polynomial

$$Q(\bar{\xi}) = P(e^{i\xi_1}, \dots, e^{i\xi_m}) = \sum_{\lambda=1}^{\Lambda} a_{\lambda} \exp [i(\bar{k}_{\lambda}, \bar{\xi})] \quad \left((u, v) \stackrel{\text{def}}{=} \sum_{j=1}^m u_{jv} j \right). \quad (4)$$

Let $U\{Q(\bar{\zeta}) = 0\}$ be the (necessarily nonempty) set of all complex $\bar{\zeta} = \bar{\xi} + i\bar{\eta}$ zeros of the polynomial $Q(\bar{\zeta})$, and let

$$d(Q) = \min_{\bar{\zeta} \in U} |\bar{\eta}| \quad \left(|\bar{\eta}| = \sqrt{\eta_1^2 + \dots + \eta_m^2} \right). \quad (5)$$

It is obvious that when $d(Q) = 0$, the polynomial (4) has real zeros. If $d(Q) > 0$, then in the strip $|\bar{\eta}| < d(Q)$ the polynomial $Q(\bar{\zeta})$ has no zeros, and therefore the function $[Q(\bar{\zeta})]^{-1}$ is analytic in this strip.

Theorem 1. Let $d = d(Q) > 0$. Then equation (2) has a solution $\mathcal{E}(\bar{n})$ satisfying, for every $\varepsilon > 0$, the estimate

$$|\mathcal{E}(\bar{n})| \leq A_{\varepsilon} \exp[-(d - \varepsilon)|\bar{n}|]. \quad (6)$$

(An analogous estimate for the Fourier transform of a function analytic in a strip was obtained in ¹.)

Proof. It is easy to verify directly that the desired solution is the Fourier coefficient of the function $[Q(\bar{\xi})]^{-1}$, i.e.

$$\mathcal{E}(\bar{n}) = (2\pi)^{-m} \int_0^{2\pi} \dots \int_0^{2\pi} [Q(\bar{\xi})]^{-1} \exp[i(\bar{n}, \bar{\xi})] d\xi_1 \dots d\xi_m.$$

This coefficient can be written in the form

$$\mathcal{E}(\bar{n}) = (2\pi i)^{-m} \oint_{C_1} \dots \oint_{C_m} z_1^{n_1-1} \dots z_m^{n_m-1} [P(z)]^{-1} dz_1 \dots dz_m, \quad (7)$$

where C_j is the unit circle in the z_j -plane. Since, under a continuous change of the radii of these circles, expression (7) does not change as long as the integration manifold does not encounter singular points of the integrand, making the substitution $z_j = \exp[i(\xi_j + i\eta_j)]$, we obtain

$$\mathcal{E}(\bar{n}) = (2\pi)^{-m} \int_0^{2\pi} \dots \int_0^{2\pi} [Q(\bar{\xi} + i\bar{\eta})]^{-1} \exp[i(\bar{n}, \bar{\xi})] d\xi_1 \dots d\xi_m \cdot \exp[-(\bar{n}, \bar{\eta})]$$

for every $|\bar{\eta}| < d$. Hence it follows that, for every fixed $\varepsilon \in (0, d)$, when $|\bar{n}| \leq d - \varepsilon$ we have

$$|\mathcal{L}(\bar{n})| \leq A_\varepsilon \exp[-(\bar{n}, \bar{\eta})].$$

Putting here

$$\bar{\eta} = |\bar{n}|^{-1}(d - \varepsilon)\bar{n},$$

we obtain (6). The theorem is proved.

Theorem 1 generalizes one of the principal results of ², where, under the assumption that the operator \mathcal{L} has a “dominant” coefficient, i.e. when

$$|a_1| + \dots + |a_{\Lambda-1}| < |a_\Lambda|, \quad (8)$$

the existence of a solution of equation (2) decreasing exponentially to zero as $|\bar{n}| \rightarrow \infty$ is proved. It is clear that under assumption (8) the polynomial (4) has no real zeros, and therefore the condition $d > 0$ of Theorem 1 is satisfied. The class of equations for which $d > 0$ is considerably broader than the class of equations with a dominant coefficient. For example, for the operator

$$Lu(\bar{n}) = a_1u(n_1, n_2) + a_2u(n_1 + 1, n_2) + a_3u(n_1, n_2 + 1) + a_4u(n_1 + 1, n_2 + 1)$$

the condition $d > 0$ has the form

$$||a_1|^2 + |a_2|^2 - |a_3|^2 - |a_4|^2| > 2|a_1a_2^* - a_3a_4^*|$$

(the asterisk denotes the complex conjugate); in particular, for real coefficients this means that

$$\left| \max_j a_j + \min_j a_j \right| > \left| \sum_{j=1}^4 a_j - \max_j a_j - \min_j a_j \right|.$$

For what follows we introduce the linear space Φ_α invariant with respect to integer shifts (α is any real number) of functions $\varphi(\bar{n})$ for which

$$\sup_{\bar{n}} [|\varphi(\bar{n})| \exp(-\alpha|\bar{n}|)] < \infty,$$

as well as the convolution operation

$$(\varphi_1 * \varphi_2)(\bar{n}) = \sum_{\bar{m}}^{\text{def}} \varphi_1(\bar{m})\varphi_2(\bar{n} - \bar{m})$$

under the assumption of absolute convergence of this series.

Lemma. If $\varphi_1 \in \Phi_\alpha$, $\varphi_2 \in \Phi_{-|\alpha|-\varepsilon}$, where $\varepsilon > 0$, then $\varphi_1 * \varphi_2$ is defined and belongs to Φ_α .

Proof follows immediately from the fact that

$$\sup_{\bar{n}} \sum_{\bar{m}} \exp[(-|\alpha| - \varepsilon)|\bar{m}| + \alpha|\bar{n} - \bar{m}| - \alpha|\bar{n}|] \leq \sum_{\bar{m}} \exp(-\varepsilon|\bar{m}|) < \infty.$$

Introduce the operator adjoint to \mathcal{L} by the formula

$$\mathcal{L}^* \varphi(\bar{n}) = \sum_{\lambda=1}^{\Lambda} a_\lambda^* \varphi(n_1 - k_{\lambda 1}, \dots, n_m - k_{\lambda m}).$$

Then, for any functions $\varphi(\bar{n})$ and $\psi(\bar{n})$,

$$\{\mathcal{L}\varphi, \psi\} = \{\varphi, \mathcal{L}^*\psi\} \quad \left(\{\varphi, \psi\} \stackrel{\text{def}}{=} \sum_{\bar{n}} \varphi(\bar{n}) \psi^*(\bar{n}) \right),$$

provided that interchange of terms after expanding all parentheses is assumed possible. The trigonometric polynomial (4) corresponding to the operator \mathcal{L}^* is equal to

$$Q^+(\bar{\xi}) = \sum_{\lambda} a_\lambda^* \exp[i(-\bar{k}_\lambda, \bar{\xi})] = [Q(\bar{\xi}^*)]^*,$$

and therefore the numbers d defined by formula (5) for the operators \mathcal{L} and \mathcal{L}^* are identical.

Theorem 2. Let the constant $d(Q) > 0$. Then for every function $\psi \in \Phi_\alpha$, where $-d < \alpha < d$, in the class Φ_α there exists one and only one solution of equation (3).

Proof. The desired solution is constructed by the formula

$$u(\bar{n}) = \sum_{\bar{m}} \mathcal{E}(\bar{n} - \bar{m}) \psi(\bar{m}) = \mathcal{E} * \psi, \quad (9)$$

where the function $\mathcal{E}(\bar{n})$ was constructed in Theorem 1. From estimate (6) and the lemma it follows that $u \in \Phi_\alpha$. To prove uniqueness of the solution, suppose that $v \in \Phi_\alpha$ satisfies equation (1), and construct, by Theorem 1, Green's function $\mathcal{E}^*(\bar{n}) = [\mathcal{E}(-\bar{n})]^*$ for the operator \mathcal{L}^* . Then, for any fixed \bar{m} ,

$$0 = \{\mathcal{L}v, \mathcal{E}^*(\bar{n} - \bar{m})\} = \{v, \mathcal{L}^* \mathcal{E}^*(\bar{n} - \bar{m})\} = v(\bar{m}),$$

i.e. $v \equiv 0$. Theorem 2 is proved.

Remark 1. By Theorem 2, Φ_α for $\alpha < d$ is a uniqueness class for the solutions of equation (1). At the same time, Φ_d is no longer such a class, since Φ_d contains the nontrivial solution of this equation

$$u(\bar{n}) = \exp[i(\bar{n}, \bar{\xi}_0)], \quad (10)$$

where $\bar{\xi}_0 = \bar{\xi}_0 + i\bar{\eta}_0$ is a solution of the equation $Q(\bar{\xi}_0) = 0$, with $|\bar{\eta}_0| = d$.

Remark 2. In the case $d = 0$ it is easy to establish a sharper uniqueness assertion: if $u(\bar{n})$ satisfies equation (1) and $\sum_{\bar{n}} |u(\bar{n})| < \infty$, then $u \equiv 0$. This follows directly from the fact that the continuous function

$$\tilde{u}(\bar{\xi}) = \sum_{\bar{n}} u(\bar{n}) \exp[-i(\bar{n}, \bar{\xi})]$$

satisfies the equation $Q(\bar{\xi})\tilde{u}(\bar{\xi}) = 0$. At the same time, the function (10) is a nonzero bounded solution of equation (1).

Corollary. For equation (1) to have a nontrivial bounded solution, it is necessary and sufficient that the trigonometric polynomial $Q(\bar{\xi})$ have real zeros.

We now formulate the Cauchy problem for equation (3) (a special case of which is (1)); for this purpose we distinguish one of the independent variables, for example the last. Let $k'_m = \min_\lambda k_{\lambda m}$, $k''_m = \max_\lambda k_{\lambda m}$, and let

$$k = k''_m - k'_m > 0$$

(if $k'_m = k''_m$, then the variable n_m is simply a parameter). Obviously, without loss of generality one may assume $k'_m = 0$. The problem

The Cauchy problem consists in finding a solution $u(\bar{n})$ of equation (3) for the values

$$-\infty < n_1, \dots, n_{m-1} < \infty, \quad N_0 \leq n_m < \infty \quad (-\infty < N_0 < \infty), \quad (11)$$

where this solution must satisfy the conditions

$$u(\bar{n}', N_0) = \varphi_1(\bar{n}'), \dots, u(\bar{n}', N_0 + k - 1) = \varphi_k(\bar{n}'),$$

$$\left(\bar{n}' \stackrel{\text{def}}{=} (n_1, \dots, n_{m-1}); -\infty < n_1, \dots, n_{m-1} < \infty \right), \quad (12)$$

where $\varphi_1(\bar{n}'), \dots, \varphi_k(\bar{n}')$ are prescribed functions.

Equation (3) can be rewritten in the form

$$\begin{aligned} \mathcal{L}'u &\equiv \sum_{k_{\lambda m}=k''_m} a_{\lambda} u(n_1 + k_{\lambda 1}, \dots, n_m + k''_m) = \\ &= - \sum_{k_{\lambda m} < k''_m} a_{\lambda} u(n_1 + k_{\lambda 1}, \dots, n_m + k_{\lambda m}) + \psi(\bar{n}). \end{aligned} \quad (13)$$

Let us denote, similarly to (4),

$$Q'(\bar{\xi}') = \sum_{k_{\lambda m}=k''_m} a_{\lambda} \exp [i(\bar{k}_{\lambda}, \bar{\xi}')], \quad (14)$$

and let the number $d' = d'(Q')$ be determined by this polynomial in the same way as the number $d(Q)$ is determined by the polynomial $Q(\bar{\xi})$ according to formula (5). If the sum in (14) contains only one term (the case of an “explicit scheme” for the solution of the Cauchy problem), then the Cauchy problem (3)–(12) has, and moreover uniquely, a solution, whatever the prescribed initial functions $\varphi_1(\bar{n}'), \dots, \varphi_k(\bar{n}')$ and the right-hand side $\psi(\bar{n})$ may be. If, however, the sum (14) contains more than one term, then the following holds.

Theorem 3. Let $d' = d'(Q') > 0$, and let, for some $a \in (-d', d')$, all the functions $\varphi_1(\bar{n}'), \dots, \varphi_k(\bar{n}')$, as well as $\psi(\bar{n}', n_m)$ ($N_0 \leq n_m < \infty$), belong to the space Φ'_a (defined analogously to Φ_a). Then the solution of the Cauchy problem (3)–(12) in the half-space (11) exists and is unique in the class Φ'_a .

Proof. Putting $n_m = N_0$ in equality (13), we obtain, for determining the sought function $u(\bar{n})$ in the subspace $n_m = N_0 + k$, the equation $L'u = \psi'(\bar{n}')$, where the known function $\psi'(\bar{n}')$ is determined by substituting the initial functions into the right-hand side of (13) and, under the conditions of the theorem, belongs to Φ'_a . Application of Theorem 2 makes it possible to construct the solution for $n_m = N_0 + k$. For $N_0 + k < n_m < \infty$ the construction is analogous. Theorem 3 is proved.

All the results presented carry over to the case of systems of linear difference equations with constant coefficients. In this case the values of the functions $u(\bar{n})$ and $\psi(\bar{n})$ are column matrices (vectors) of order ≥ 1 , and the coefficients a_{λ} in the operator \mathcal{L} are square matrices of the same order. The Green matrix is defined by the same formula (7), and the solution of the system (3) is expressed by formula (9). The numbers d and d' appearing in the formulations of the theorems are determined through the trigonometric polynomials $\det Q(\bar{\xi})$ and $\det Q'(\bar{\xi}')$. (To carry over the assertions of the second paragraph of the paper it is necessary that certain matrices a_{λ} , which are easy to indicate, be nonsingular; the same applies to the solvability of the Cauchy problem in the case of an explicit scheme.)

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Note: Figure translations are in progress. See original paper for figures.

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