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Abstract

Full Text

MATHEMATICS

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THE GRID METHOD FOR SOME PROBLEMS OF MATHEMATICAL PHYSICS

(Presented by Academician V. I. Smirnov on 23 V 1964)

In the well-known estimates of the rate of convergence of the grid method in the Dirichlet and Neumann problems (¹⁻⁹) for a second-order elliptic equation, the assumption is used that continuous fourth derivatives of the solution exist (in most works, up to the boundary of the domain under consideration); moreover, the constants entering into the estimate depend on the maximum of the moduli of these derivatives. For the particular case—the Laplace equation in a rectangular domain—estimates are given (uniform in the whole domain or in some subdomain of it) into which enter only quantities explicitly depending on the data of the problem (¹⁰⁻¹³). In the works listed, the matrix of the system of grid equations is computed from the values at the grid nodes of the coefficients of the differential operator and thus does not take into account the behavior of the coefficients in a neighborhood of the nodes under consideration.

The aim of the present work is, in the case of the Dirichlet and Neumann problems, to construct a system of grid equations using averaging of the coefficients of the differential operator and to estimate the rate of convergence of the method in terms of the data of the problem, assuming the following about the smoothness of the solution: the solution of the problem has generalized second derivatives summable with the square.

In a bounded n -dimensional domain Ω , consider the equation

$$A_0 u \equiv - \sum_{i,k=1}^n \frac{\partial}{\partial x_i} \left(a_{ik} \frac{\partial u}{\partial x_k} \right) + au = f, \quad f \in L_2(\Omega), \quad (1)$$

where a_{ik} and a are measurable and bounded functions in the domain Ω ,

$$\sum_{i,k=1}^n a_{ik}(x) \xi_i \xi_k \geq \gamma \sum_{i=1}^n \xi_i^2, \quad \gamma = \text{const} > 0,$$

and the quadratic form

$$[u, u] = \int_{\Omega} \left(\sum_{i,k=1}^n a_{ik} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} + au^2 \right) d\omega$$

is positive definite:

$$[u, u] \geq \mu_0(u, u), \quad \mu_0 = \text{const} > 0, \quad (u, v) = \int_{\Omega} uv \, d\omega.$$

The closure of the linear manifold of continuously differentiable functions with compact support in Ω in the norm $\| \cdot \|$, $\|u\| = \sqrt{[u, u]}$, will be denoted by H_{A_0} . By a solution of the Dirichlet problem with zero boundary condition for equation (1) we shall mean (see (14)) the solution of the problem of minimizing the functional

$$F(u) = [u, u] - 2(u, f), \quad f \in L_2(\Omega), \quad u \in H_{A_0}. \quad (2)$$

1. On the domain Ω impose a cubic grid of step h , with edges parallel to the coordinate axes, and denote by $|\Omega_h|$ the smallest union of cubes containing Ω .

Let P be a certain node of the grid; by Π_P denote that cube of our grid which turns out to lie in the first octant if the coordinate frame $(x_1, \dots, x_n) = \{x_i\}$, parallel to itself, is translated to the point P .

Construct an open ball of radius $\frac{1}{2}\varepsilon\sqrt{n}$ with center at a certain point x of the domain Ω . The union of all such balls, as the point x ranges over the domain Ω , will be denoted by $\Omega_{(\varepsilon)}$. The set of those points of the domain Ω whose distance to the boundary of Ω is not less than h will be denoted by Ω^h .

Let $u(P)$ be a function defined at the nodes of the mesh, $\alpha = (\alpha_1, \dots, \alpha_n)$ a vector of the n -dimensional space R_n , $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, $i = 1, \dots, n$.

Introduce the notation:

$$D_h^0 u(P) = \overline{D}_h^0 u(P) = u(P), \quad \theta = (0, \dots, 0), \quad \alpha = (\alpha_1, \dots, \alpha_n) \in A,$$

$$D_h^{e_i} u(P) = \frac{1}{h} [u(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - u(x_1, \dots, x_n)],$$

$$\overline{D}_h^{e_i} u(P) = \frac{1}{h} [u(x_1, \dots, x_n) - u(x_1, \dots, x_{i-1}, x_i - h, x_{i+1}, \dots, x_n)],$$

$$D_h^\alpha u(P) = D_h^{\alpha_1 e_1} \dots D_h^{\alpha_n e_n} u(P), \quad \overline{D}_h^\alpha u(P) = \overline{D}_h^{\alpha_1 e_1} \dots \overline{D}_h^{\alpha_n e_n} u(P),$$

where A is the set of all n -dimensional vectors whose components are equal to zero or one, and P is a mesh node with coordinates (x_1, \dots, x_n) .

Let the boundary S of the domain Ω be twice continuously differentiable, and let the number h_0 be such that, for $h < h_0$, the correspondence

$$x = y + tm(y)$$

between points of the strip $\Omega \setminus \Omega^{2h}$ and pairs (y, t) ($y \in S$, t is the distance from the point x to the surface S) is locally one-to-one, where $m(y)$ is the unit inward normal to the surface S at the point y . In what follows we shall assume $h < h_0$.

Denote

$$\chi_h(x) = \begin{cases} 0, & x \in \Omega^h, \\ \frac{t-h}{h}, & x = y + tm(y), \quad h \leq t < 2h, \quad y \in S, \\ 1, & x \in \Omega^{2h}. \end{cases}$$

The function

$$\tilde{u}(x) = \chi_h(x) \sum_{\alpha \in A} D_h^\alpha u(P_0) w_\alpha(x, P_0), \quad x \in \Pi_{P_0}, \quad (3)$$

where

$$w_\alpha(x, P_0) = (x_1 - x_1^0)^{\alpha_1} \cdots (x_n - x_n^0)^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n), \\ P_0 = (x_1^0, \dots, x_n^0), \quad x = (x_1, \dots, x_n),$$

is continuous, piecewise continuously differentiable, and finite in Ω . Clearly, $\tilde{u} \in \dot{W}_2^1(\Omega)$, and the functional (2) on the finite-dimensional subspace H_h of functions of the form (3) can be written in the following form:

$$F(\tilde{u}) = \sum_{\Pi_{P_0} \subset |\Omega_h|} \left\{ \sum_{\alpha, \beta \in A} D_h^\alpha u(P_0) D_h^\beta u(P_0) \int_{\Pi_{P_0}} \left(\sum_{i, j=1}^n a_{ij} \frac{\partial \chi_h w_\alpha}{\partial x_i} \frac{\partial \chi_h w_\beta}{\partial x_j} + a w_\alpha w_\beta \chi_h^2 \right) d\omega - 2 \sum_{\alpha \in A} D_h^\alpha u(P_0) \int_{\Pi_{P_0}} \dots \right\}$$

Let us consider the mesh functions \tilde{A}_α^β and \tilde{f}^α . For $\Pi_{P_0} \subset |\Omega_h|$,

$$\tilde{A}_\alpha^\beta(P_0) = \frac{1}{h^n} \int_{\Pi_{P_0}} \left\{ \sum_{i, j=1}^n a_{ij}(x) \frac{\partial w_\alpha(x, P_0) \chi_h(x)}{\partial x_i} \frac{\partial w_\beta(x, P_0) \chi_h(x)}{\partial x_j} + a(x) w_\alpha(x, P_0) w_\beta(x, P_0) \chi_h^2(x) \right\} d\omega, \\ \tilde{f}^\alpha(P_0) = \frac{1}{h^n} \int_{\Pi_{P_0}} f(x) w_\alpha(x, P_0) \chi_h(x) d\omega,$$

and at the remaining nodes set them equal to zero.

Then the condition for a minimum of the functional (2) on the subspace H_h can be written in the form

$$\sum_{\alpha, \beta \in A} (-1)^{|\alpha|} \overline{D_h^\alpha} \left(\tilde{A}_\beta^\alpha(P) D_h^\beta u(P) \right) = \sum_{\alpha \in A} (-1)^{|\alpha|} \overline{D_h^\alpha} \tilde{f}^\alpha(P), \quad (4)$$

$$|\alpha| = \sum_{i=1}^n |\alpha_i|, \quad P \in |\Omega_h|, \quad P \text{ is a grid node.}$$

The unique solvability of system (4) follows easily from the way in which it was obtained.

Example. For the Laplace operator, at all nodes lying in the domain $\Omega^{h\sqrt{n}}$, one obtains a 9-point finite-difference scheme approximating the operator on four times continuously differentiable functions with order h^2 .

2. In this section we shall assume that $\Omega = |\Omega_h|$. Let $g_h(x)$ be the averaging of the function $g(x)$ over the cube Π : $-h/2 \leq t_i \leq h/2$ ($i = 1, \dots, n$),

$$g_h(x) = \frac{1}{h^n} \int_{\Pi} g(x+t) dt.$$

Denote

$$\omega_{L_2(\Omega)} \left(g, \frac{h}{2} \right) = \sup_{t \in \Pi} \|g(x+t) - g(x)\|_{L_2(\Omega)}, \quad g \in L_2(\Omega_{(\varepsilon)}), \quad h < \varepsilon,$$

$$\omega_{W_2^1(\Omega)} \left(g, \frac{h}{2} \right) = \sup_{t \in \Pi} \|g(x+t) - g(x)\|_{W_2^1(\Omega)}, \quad g \in W_2^1(\Omega_{(\varepsilon)}), \quad h < \varepsilon.$$

Theorem 1. Let $g \in W_2^1(\Omega_{(\varepsilon)})$, $\varepsilon > 0$,

$$\tilde{u}(x) = \sum_{\alpha \in A} D_h^\alpha u(P_0) \omega_\alpha(x, P_0), \quad x \in \Pi_{P_0}, \quad u(P_0) = g_{hhh}(P_0), \quad h < \frac{1}{5}\varepsilon.$$

Then the inequality

$$\|\tilde{u} - g\|_{W_2^1(\Omega)} \leq K \omega_{W_2^1(\Omega)} \left(g, \frac{5h}{2} \right) \quad (5)$$

holds, where the number K is positive and does not depend on g or h .

3. We note that under the conditions formulated earlier there exist [14] constants ν_0 and ν_1 such that

$$\nu_0 \|u\|_{W_2^1(\Omega)} \leq |u| \leq \nu_1 \|u\|_{W_2^1(\Omega)}, \quad u \in H, \quad \nu_i > 0, \quad i = 0, 1.$$

Theorem 2. Suppose that in the domain Ω , with twice continuously differentiable boundary, the Dirichlet problem with zero boundary condition is posed for equation (1).

Then the sequence of approximate solutions \tilde{u}_*

$$\tilde{u}_*(x) = \chi_h(x) \sum_{\alpha \in A} D_h^\alpha v_*(P_0) \omega_\alpha(x, P_0), \quad x \in \Pi_{P_0}, \quad (6)$$

where $v_*(P)$ is the solution of equation (4), converges as $h \rightarrow 0$ to the exact solution u^* in the metric of $W_2^1(\Omega)$, and the estimate

$$\|u^* - \tilde{u}_*\|_{W_2^1(\Omega)} \leq \frac{\nu_1}{\nu_0} (K + L) \omega_{W_2^1(\Omega)} \left(u^*, \left(\frac{5}{2} + 8\sqrt{n} \right) h \right)$$

holds

$$(u^*(x) = 0 \quad \text{for } x \in \bar{\Omega}),$$

where the constant K is taken from (5), and L depends only on the domain Ω .

Theorem 3. Suppose the conditions of Theorem 2 are satisfied. Assume also that the solution u^* of the problem under consideration lies in $W_2^2(\Omega)$, and that the inequality

$$\|u\|_{W_2^2(\Omega)} \leq m \|Au\|_{L_2(\Omega)}, \quad u \in W_2^2(\Omega) \cap H_A. \quad (7)$$

holds.

Then the following estimate holds for the closeness of the approximate solution to the exact solution:

$$\|\tilde{u}_* - u^*\|_{W_2^1(\Omega)} \leq \frac{\nu_1}{\nu_0} mC (K + L) h^{1/2} \|f\|_{L_2(\Omega)},$$

where L and C are determined by the domain, and K is the constant from (5).

Remark. Sufficient conditions for the fulfillment of assumption (7) are formulated in ¹⁵.

4. In the Dirichlet problem posed in Sec. 1, domains Ω of the following special form are also considered: $\Omega = |\Omega_h|$.

If one considers the finite-dimensional space H_h of functions of the form \tilde{u} , defined in Ω and equal to zero on the boundary S of the domain Ω , then, just as in Sec. 1, for the mesh function $u(P)$ at the interior nodes of the domain Ω we obtain the equation

$$\sum_{\alpha, \beta \in A} (-1)^{|\alpha|} \overline{D}_h^\alpha (A_\alpha^\beta(P) D_h^\beta u(P)) = \sum_{\alpha \in A} (-1)^{|\alpha|} \overline{D}_h^\alpha f^\alpha(P). \quad (8)$$

In this case assertions analogous to Theorems 2 and 3 are valid. The matrix of system (8) is positive definite, and the lower bound of its smallest eigenvalue is the number $\mu_0/3^n$.

5. Assertions analogous to those of Secs. 1, 3, and 4 are also valid in the case where one considers an admissible variational formulation of the Dirichlet (or Neumann) problem for a strongly elliptic system of differential equations with derivatives of second order. Here, in particular, we rely on an analogue of inequality (7) for elliptic systems; sufficient conditions for its fulfillment are formulated in ¹⁸.

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CITED LITERATURE

- ¹ S. Gerschgorin, *Zs. angew. Math. u. Mech.*, **10**, 373 (1930).
- ² W. Wasow, *Zs. angew. Math. u. Phys.*, **6**, 81 (1955).
- ³ W. Wasow, *Quart. Appl. Math.*, **15**, 53 (1957).
- ⁴ P. Laasonen, *Ann. Acad. Sci. Fenn.*, AI, 246, 1 (1957).
- ⁵ P. Laasonen, *J. Assoc. Comput. Mach.*, **5**, 32 (1958).
- ⁶ V. I. Lebedev, *DAN*, **128**, No. 4 (1959).
- ⁷ V. I. Lebedev, *DAN*, **132**, No. 5 (1960).
- ⁸ V. I. Lebedev, *DAN*, **134**, No. 2 (1960).
- ⁹ E. A. Volkov, *Computational Mathematics*, **1**, 34 (1957).
- ¹⁰ W. Wasow, *J. Res. Nat. Bur. Standards*, **48**, 345 (1952).
- ¹¹ J. L. Walsh, D. Young, *J. Res. Nat. Bur. Standards*, **51**, 343 (1953).
- ¹² J. L. Walsh, D. Young, *J. Math. Phys.*, **33**, 80 (1954).
- ¹³ J. H. Giese, *J. Assoc. Comput. Mach.*, **4**, 298 (1958).
- ¹⁴ S. G. Mikhailin, *The Problem of the Minimum of a Quadratic Functional*, 1952.
- ¹⁵ O. A. Ladyzhenskaya, *Mixed Boundary-Value Problem for a Hyperbolic Equation*, 1953.
- ¹⁶ B. M. Babich, *UMN*, **8**, issue 2 (54), 111 (1953).

¹⁷ S. M. Nikol' skii, *Matem. sborn.*, **40** (82), No. 2 (1956).

¹⁸ O. V. Guseva, *DAN*, **102**, No. 6 (1955).

Note: Figure translations are in progress. See original paper for figures.

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