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Abstract

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On the Concept of a Minimal Scale of Spaces

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The present paper contains a further development of a number of questions in the theory of scales of Banach spaces (see ($^{1-5}$)).

1. Let us consider a family of Banach spaces E_α $(0 \leqslant \alpha \leqslant 1)$ possessing the following property:

I. For $\alpha < \beta$, the space E_β is densely embedded in E_α and

$$\|x\|_{E_\alpha} \leqslant \|x\|_{E_\beta} \qquad (x \in E_\beta).$$

Lemma 1. If for the family E_α there exists an element $e \in E_1$ for which $\|e\|_{E_1} = \|e\|_{E_0} = 1$, then for each element $x_0 \in E_1$ one can construct a linear operator A, acting in the space E_1, such that

$$\|A\|_{E_\alpha} = \|x_0\|_{E_\alpha}.$$

Proof. By the Hahn–Banach theorem there exists a linear functional $f(x)$ from E_0' for which $\|f\|_{E_0'} = f(e) = 1$. Since $\|f\|_{E_1'} \leqslant \|f\|_{E_0'} = 1$ and $\|e\|_{E_1} = 1$, it follows that $\|f\|_{E_1'} = 1$, and consequently $\|f\|_{E_\alpha'} = 1$ $(0 \leqslant \alpha \leqslant 1)$. For any $x_0 \in E_1$ construct the linear operator A by the formula

$$A(x) = x_0 f(x).$$

Then

$$\|A\|_{E_\alpha} = \sup_{x \in E_1} \frac{\|x_0 f(x)\|_{E_\alpha}}{\|x\|_{E_\alpha}} = \|x_0\|_{E_\alpha} \sup_{x \in E_1} \frac{|f(x)|}{\|x\|_{E_\alpha}} = \|x_0\|_{E_\alpha}.$$

Definition 1. We shall say that a family of Banach spaces E_α $(0 \leqslant \alpha \leqslant 1)$ has the normal (strict) interpolation property with respect to a family of spaces F_α $(0 \leqslant \alpha \leqslant 1)$ if, for every bounded linear operator A acting from the spaces E_0 and E_1, respectively, into the spaces F_0 and F_1, it follows that it acts from any space E_α into the space F_α $(0 \leqslant \alpha \leqslant 1)$, and its norm $\|A\|_{E_\alpha \to F_\alpha}$ satisfies the inequality

$$\|A\|_{E_\alpha \to F_\alpha} \leqslant \|A\|^{1-\alpha}_{E_0 \to F_0} \|A\|^{\alpha}_{E_1 \to F_1}$$

(is a logarithmically convex function of α).

From the lemma it follows:

Corollary. Let the family E_α $(0 \leqslant \alpha \leqslant 1)$ possess the strict interpolation property with respect to itself; then this family forms a normal scale of spaces (see $(^2)$).

Let E_α and F_α be two families of Banach spaces possessing property I, and let E'_α and F'_α be the families of spaces conjugate to them. If the family F'_α possesses the strict interpolation property with respect to the family E'_α, then, as is not hard to prove by passing to the adjoint operator, the family E_α possesses the strict interpolation property with respect to the family F_α.

2. Let E_α $(0 \leq \alpha \leq 1)$ be a continuous normal scale. Introduce on the space E'_0, conjugate to E_0, the family of norms $f_{E'_\alpha}$. The completion of the space E'_0 with respect to the norm $\|f\|_{E'_\alpha}$ will be denoted by $\widetilde{E}_{-\alpha}$.

Definition 2. A continuous normal scale E_α $(0 \leq \alpha \leq 1)$ will be called **regular** if the spaces $\widetilde{E}_{-\alpha}$ form a normal scale on the interval $(-1, 0)$.

The scale $\widetilde{E}_{-\alpha}$ $(0 \leq \alpha \leq 1)$ is called **conjugate** to the scale E_α; $E_{-\alpha}$ is a continuous normal scale.

A continuous normal scale need not be regular. As an example one may give the following scale: let $E_\alpha = E_0$ for $0 \leq \alpha \leq \frac{1}{2}$, and $E_\alpha = F_\alpha$, where F_α is an arbitrary nontrivial continuous normal scale on the interval $[\frac{1}{2}, 1]$, with $F_{1/2} = E_0$.

Theorem 1. *The maximal scale* $^{(2)}$*, constructed from two normally embedded Banach spaces* E_0 *and* E_1*, is a regular scale.*

3. Let E_0 and E_1 be two related spaces. Introduce on the space E_1 the family of norms

$$\|x\|_{E^0_\alpha} = \sup_{f \in E'_0} \frac{|f(x)|}{\|f\|_0^{1-\alpha} \|f\|_1^{\alpha}}. \tag{1}$$

As shown in $^{(3)}$, the completions of the space E_1 with respect to the norms (1) form a continuous normal scale E^0_α connecting E_0 and E_1. We shall call this scale **minimal**. This term is connected with the following theorem.

Theorem 2. *The minimal scale* E^0_α *is majorized by any regular scale* E_α *connecting the spaces* E_0 *and* E_1*, i.e.*

$$\|x\|_{E^0_\alpha} \leq \|x\|_{E_\alpha} \qquad (x \in E_1;\ 0 \leq \alpha \leq 1).$$

Proof. Let $x \in E_1$. The spaces E_α and E_1 are related; therefore, from the proof of Theorem 1 in $^{(3)}$ it follows that

$$\|x\|_{E_\alpha} = \sup_{f\in E_0'} \frac{|f(x)|}{\|f\|_{E_\alpha'}}.$$

By virtue of the regularity of the scale E_α,

$$\|f\|_{E_\alpha'} \le \|f\|_{E'0}^{1-\alpha}\|f\|_{E'1}^{\alpha}, \qquad (f \in E_0').$$

Then

$$\|x\|_{E_\alpha} \sup_{f\in E_0'} \frac{|f(x)|}{\|f\|^{1-\alpha}_{E'0}\|f\|^{\alpha}_{E'1}} = \|x\|_{E^0_\alpha}.$$

The theorem is proved.

Theorem 3. *The minimal scale has the normal interpolation property with respect to any other minimal scale.*

Proof. Let E^0_α be a minimal scale connecting the spaces E_0 and E_1, and let F^0_α be a minimal scale constructed from the spaces F_0 and F_1. If a linear operator A, acting in E_1, satisfies the conditions

$$\|Ax\|_{F_0} \le C_0\|x\|_{E_0}, \qquad \|Ax\|_{F_1} \le C_1\|x\|_{E_1},$$

then the conjugate operator A^* maps the space F_0' into E_0' and F_1' into E_1', and the inequalities

$$\|A^*f\|_{E_0'} \le C_0\|f\|_{F_0'}, \qquad (f \in F'),$$

$$\|A^*f\|_{E_1'} \le C_1\|f\|_{F_1'}.$$

Let the element $x \in E_1$; then

$$\|Ax\|_{F_\alpha} = \sup_{f\in F_0'} \frac{|f(Ax)|}{\|f\|^{1-\alpha}_{F_0'}\|f\|^{\alpha}_{F_1'}} = \sup_{f\in F_0'} \frac{|A^*f(x)|}{\|f\|^{1-\alpha}_{F_0'}\|f\|^{\alpha}_{F_1'}} \le C_0^{1-\alpha}C_1^{\alpha} \sup_{f\in F_0'} \frac{|A^*f(x)|}{\|A^*f\|^{1-\alpha}_{E_0'}\|A^*f\|^{\alpha}_{E_1'}} \le$$

$$\le C_0^{1-\alpha}C_1^{\alpha} \sup_{g\in E_0'} \frac{|g(x)|}{\|g\|^{1-\alpha}_{E_0'}\|g\|^{\alpha}_{E_1'}} = C_0^{1-\alpha}C_1^{\alpha}\|x\|_{E_\alpha};$$

the theorem is proved.

Corollary. *Every regular scale has the normal interpolation property with respect to any minimal scale.*

4. Let E_α $(0 \le \alpha \le 1)$ be a continuous normal scale. Consider the family of Banach spaces $E_{-\alpha}$. Each element $x \in E_1$ gives rise in the natural way to a continuous linear functional $x(f) = f(x)$ $(f \in E_{-1})$ on the space $\widetilde{E}_{-1}$. From the proof of Theorem 1 it follows that the norm of this functional $x(f)$ coincides with the norm $\|x\|_{E_1}$. If all continuous linear functionals on the space $\widetilde{E}_{-1}$ are exhausted by functionals of the form $x(f)$ $(x \in E_1,\ f \in \widetilde{E}_{-1})$, then we shall call the scale E_α **reflexive**. For a reflexive scale, in a certain sense, the second conjugate scale coincides with the original one.

Lemma 2. *If a minimal scale E_α is regular and reflexive, then the scale conjugate to it forms a maximal scale joining the spaces $\widetilde{E}_0$ and $\widetilde{E}_{-1}$.*

Theorem 4. *Every regular scale has the strict interpolation property with respect to a minimal scale whose conjugate is maximal.*

5. For the spaces $L_1(0,1)$ and $L_\infty(0,1)$, a maximal scale was constructed in [4]. It consists of Lorentz spaces S_α, in which the norm is defined by the formula

$$\|x\|_{S_\alpha} = \alpha \int_0^1 t^{\alpha-1} x^*(t)\, dt,$$

where $x^*(t)$ is the rearrangement of the function $|x(t)|$ in nonincreasing order.

The scale S_α^* is conjugate to the scale M_α^0, where the space M_α^0 consists of all measurable functions for which

$$\|x\|_{M_\alpha^0} = \sup \frac{\displaystyle\int_E |x(t)|\, dt}{(\operatorname{mes} E)^\alpha} < \infty,$$

$$\lim_{\operatorname{mes} E \to 0} \frac{\displaystyle\int_E |x(t)|\, dt}{(\operatorname{mes} E)^\alpha} = 0 \tag{5}$$

The scale M_α^0 is a minimal scale joining the spaces $L_1(0,1)$ and $L_\infty(0,1)$.

It follows from Theorem 4 that every regular scale has the strict interpolation property with respect to the scale M_α^0.

We note that regular scales include the scales of the spaces L_p, M_α^0, S_α, Hilbert scales, etc. Every minimal scale has the normal interpolation property with respect to the scale M_α^0.

Consider now the space C_α^0 $(0 \le \alpha \le 1)$, consisting of all functions $x(t)$, $0 \le t \le 1$, for which

$$\lim_{t \to \tau} \frac{|x(t) - x(\tau)|}{|t-\tau|^\alpha} = 0 \qquad (t, \tau \in [0,1]).$$

Identifying all functions that differ by a constant, introduce in the space C_α^0 the norm

$$\|x\|_{C_\alpha^0} = \sup_{0 \le t, \tau \le 1} \frac{|x(t) - x(\tau)|}{|t-\tau|^\alpha} \qquad (0 \le \alpha \le 1).$$

The spaces C_α^0 $(0 \le \alpha \le 1)$ form a continuous normal scale (^{2,6}), joining the spaces $C(0,1)$ and $C_1(0,1)$. If, from these spaces, one constructs the minimal scale E_α^0, then the norm

$$\|x\|_{C_\alpha^0} \le \|x\|_{E_\alpha^0} \qquad (x \in C_1(0,1),\ 0 \le \alpha \le 1). \tag{2}$$

The scale $\Phi_\alpha = \widetilde{E}_{-\alpha}$, conjugate to the scale C_α^0 $(0 \le \alpha \le 1)$, consists of spaces of absolutely additive functions on the interval $[0,1]$, considered in (^{7}).

It is shown there that the space conjugate to Φ_α^0 $(0 < \alpha \le 1)$ is the Hölder space C_α.

The family of Hölder spaces, as shown in (^{8}), has the strict interpolation property with respect to itself. As was indicated, it follows from this that the family of spaces Φ_α also has the strict interpolation property with respect to itself and therefore (Lemma 1) forms a normal scale. Thus, the scale C_α^0 is proper. By Theorem 2, the inequality inverse to (2) is valid, i.e. the scale C_α^0 coincides with the minimal scale constructed from the spaces $C[0,1]$ and $C_1(0,1)$.

The scale C_α^0 is reflexive; therefore Lemma 2 on the maximality of the conjugate scale is valid for it.

Corollary. *Every proper scale (minimal scale) has the strict (normal) interpolation property with respect to the scale* C_α^0.

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