# MATHEMATICS

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## Abstract

## Full Text

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# ONE-SIDED INEQUALITIES OF CHEBYSHEV TYPE

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1. Numerous generalizations and refinements of Chebyshev' s inequality are known, according to which

$$\mathbf{P}\{|X| \geqslant \varepsilon\} \leqslant \frac{E(X^2)}{\varepsilon^2} \tag{1}$$

for any random variable $X$ and any $\varepsilon > 0$. (Surveys of these results are contained in ([1,2]).) Here it is usually assumed that the random variables under consideration have finite moments of some order. For example, inequality (1) is nontrivial only when $E(X^2) < \infty$. Such conditions impose restrictions on the behavior of the distribution function $F(x)$ of the random variable $X$ on the whole real line. At the same time, one can obtain inequalities for the probabilities of events of the form $X \geqslant \varepsilon$ or $X \leqslant -\varepsilon$ under conditions concerning the behavior of the distribution function $F(x)$ only on one of the half-lines. These inequalities may be nontrivial also in the case when $E|X|^r = \infty$ for every $r \geqslant 1$.

2. We shall give the following direct generalization of Chebyshev' s inequality.

**Theorem 1.** Let $X$ be a random variable with arbitrary distribution function $F(x) = \mathbf{P}(X < x)$, and let $c$ be any nonnegative constant. Let $g(x)$ be a nonnegative even function, defined in the domain $|x| \geqslant c$ and nondecreasing for $x \geqslant c$. Further, let $t$ be any constant satisfying the conditions $t > c$, $g(t-0) > 0$. Then

$$\mathbf{P}(X \geqslant t) \leqslant \frac{1}{g(t-0)} \int_{c+0}^{+\infty} g(x)\, dF(x), \tag{2}$$

$$\mathbf{P}(X \leqslant -t) \leqslant \frac{1}{g(t-0)} \int_{-\infty}^{-c-0} g(x)\, dF(x). \tag{3}$$

**Proof.** The proof of Theorem 1 is as simple as the proof of Chebyshev' s inequality. Namely,

$$\int_{c+0}^{+\infty} g(x)\,dF(x) \geqslant \int_{t-0}^{+\infty} g(x)\,dF(x) \geqslant g(t-0)\int_{t-0}^{+\infty} dF(x) = g(t-0)\mathbf{P}(X \geqslant t).$$

This implies (2). Inequality (3) is proved in the same way.

In order for inequality (2) to hold, it is sufficient to require that $g(x)$ be a nonnegative function nondecreasing for $x \geqslant c$.

From (2) and (3), for $c = 0$, it follows that for any $t > 0$

$$\mathbf{P}(|X| \geqslant t) \leqslant \frac{1}{g(t-0)} \int_{-\infty}^{+\infty} g(x)\,dF(x), \tag{4}$$

where the prime means that the point $x = 0$ is excluded from the domain of integration. It is easy to indicate such distribution laws for which, for a given function $g$, inequality (4) is satisfied trivially (i.e., the right-hand side of (4) is not less than one), while one of the inequalities (2) and (3) is nontrivial.

Putting in Theorem 1 $g(x) = |x|^r$ $(r > 0)$ and $c = 0$, we obtain the following result.

**Theorem 2.** *Let $X$ be a random variable with arbitrary distribution function $F(x)$. For any $t > 0$ and $r > 0$ we have*

$$\mathbf{P}(X \geq t) \leq \frac{1}{t^r} \int_{+0}^{+\infty} x^r\,dF(x), \tag{5}$$

$$\mathbf{P}(X \leq -t) \leq \frac{1}{t^r} \int_{-\infty}^{-0} |x|^r\,dF(x). \tag{6}$$

Inequalities (2)–(6) become equalities for some special distributions. For example, if $X$ has two possible values $-1$ and $+1$, to each of which corresponds probability $1/2$, then $\mathbf{P}(X \geq 1) = \mathbf{P}(X \leq -1) = 1/2$, and the right-hand sides of inequalities (5) and (6) for $t = 1$ are also equal to $1/2$. More interesting are extremal distributions with infinite moments. Let the random variable $X$ have distribution function $F(x)$, which for $x \leq 0$ has derivative $F'(x) = C(1 + |x|^{1+\delta})^{-1}$, where $\delta$ is an arbitrarily small positive constant, and the other constant $C$ is determined from the condition $F(0) = 1/2$. Let, further, $F(1+0) - F(1-0) = \mathbf{P}(X = 1) = 1/2$. Obviously, $E|X|^r = \infty$ for any $r \geq \delta$. Inequality (5) for $t = 1$ takes the form $\mathbf{P}(X \geq 1) \leq 1/2$.

3. We now give a one-sided analogue of the inequality of S. N. Bernstein [3].

**Theorem 3.** *Let $X_1, X_2, \dots, X_n$ be mutually independent random variables. Put*

$$m_{kj} = \int_{c}^{+\infty} x^k\,dF_j(x),$$

*where $F_j(x)$ is the distribution function of the random variable $X_j$. Suppose that for some value $c$ in the range $-\infty \leq c \leq 0$ there exists a positive constant $H$ such that*

$$|m_{kj}| \leq \frac{k!}{2} H^{k-2} m_{2j} \tag{7}$$

*for all integers $k \geq 2$ and $j = 1, \ldots, n$. Suppose, further, that $m_{2j} < \infty$ for the same $c$ and $j = 1, \ldots, n$. Then*

$$\mathbf{P}\left(\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} m_{1j} \geq x M_n\right) \leq e^{-x^2/4} \tag{8}$$

*for $0 < x \leq \dfrac{M_n}{H}$, where $M_n^2 = \sum_{j=1}^{n} m_{2j}$.*

**Proof.** The proof of this theorem is a modification of the proof of Bernstein' s inequality. Choose the constant $\varepsilon$ subject to the condition $0 < \varepsilon \leq \dfrac{1}{2H}$. Put

$$R_j = E e^{\varepsilon X_j}, \qquad r_j = \int_c^{\infty} e^{\varepsilon x}\, dF_j(x) \quad (j = 1, \ldots, n).$$

Expanding $e^{\varepsilon x}$ in a power series, we obtain

$$r_j = \sum_{p=0}^{\infty} \frac{\varepsilon^p}{p!} \int_c^{\infty} x^p\, dF_j(x) = \sum_{p=0}^{\infty} \frac{\varepsilon^p}{p!} m_{pj}.$$

The series $\sum_{p=2}^{\infty} \frac{\varepsilon^p}{p!} m_{pj}$ is majorized, by virtue of (7), by the series $\sum_{p=0}^{\infty} (\varepsilon H)^p \frac{\varepsilon^2}{2} m_{2j}$, whose sum is equal to $\frac{\varepsilon^2 m_{2j}}{2(1-\varepsilon H)} \leqslant \varepsilon^2 m_{2j}$. Therefore

$$\begin{aligned} R_j = \int_{-\infty}^{c} e^{\varepsilon x}\, dF_j(x) + r_j \leqslant \int_{-\infty}^{c} dF_j(x) + r_j \leqslant 1 + \varepsilon m_{1j} + \varepsilon^2 m_{2j} \\ \leqslant \exp\{\varepsilon m_{1j} + \varepsilon^2 m_{2j}\}. \end{aligned} \tag{9}$$

Applying Chebyshev' s inequality, we obtain

$$\mathbf{P}\left(\exp\left\{\varepsilon \sum_{j=1}^{n} X_j\right\} \geqslant e^{x^2/4} R_1 \cdots R_n\right) \leqslant e^{-x^2/4}.$$

Taking (9) into account, we obtain from this

$$\mathbf{P}\left(\varepsilon \sum_{j=1}^{n} X_j \geqslant \frac{x^2}{4} + \varepsilon \sum_{j=1}^{n} m_{1j} + \varepsilon^2 M_n^2\right) \leqslant e^{-x^2/4}. \tag{10}$$

If we take here $\varepsilon = x/2M_n$, then for $0 < x \leqslant M_n/H$ we shall have $0 < \varepsilon \leqslant 1/2H$. From (10), after small computations, we obtain (8).

For $c = -\infty$ and $EX_j = 0$ $(j = 1, \dots, n)$, Theorem 3 reduces to Bernstein' s theorem. A result close to Theorem 3 was obtained by V. M. Zolotarev[4] under the additional condition $E|X_j| < \infty$ $(j = 1, \dots, n)$.

It is of interest to seek one-sided analogues of other probability inequalities of Chebyshev type.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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