



Soviet-era science, translated into English

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1964

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Abstract

Full Text

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REGULAR SOLUTIONS OF MONGE–AMPÈRE EQUATIONS

(Presented by Academician V. I. Smirnov on 24 II 1964)

1. The present paper is devoted to new results on the existence of solutions of the first boundary-value problem, regular in a closed convex domain, for the equation

$$rt - s^2 = \varphi(x, y, z, p, q), \quad \varphi(x, y, z, p, q) > 0. \quad (1)$$

In earlier works ^(1–3) the main attention was devoted to the investigation of differential properties of generalized solutions of equation (1) only inside the domain. The results obtained there in passing, concerning regular solutions of equation (1) in a closed domain, have the character of rough sufficient conditions.

The results presented below on solutions of the first boundary-value problem, regular in a closed domain, have, in a certain sense, a definitive character. In a number of cases they give necessary and sufficient conditions for the solvability of the first boundary-value problem for equation (1) in the spaces $C^{m,\alpha}$.

Further, in the papers ^(1–3) an essential role was played by the uniqueness theorem for the Dirichlet problem. In connection with this, the condition $\varphi_z(x, y, z, p, q) \geq 0$ was imposed on the function $\varphi(x, y, z, p, q)$, ensuring uniqueness of the solution of the Dirichlet problem in the class of solutions whose convexity is turned downward.

Below we free ourselves from the restriction $\varphi_z \geq 0$. This makes it possible to establish the existence of many regular solutions of the first boundary-value problem for equation (1).

Throughout the entire paper, equation (1) is considered in a domain Ω , bounded by a closed convex curve Γ .

2. Here we introduce a number of concepts that are constructed from the boundary condition of the first boundary-value problem.

Throughout the entire paper we shall assume that the curve Γ has the following properties:

- a) the curvature at all points of Γ is not less than $\kappa_0 = \text{const} > 0$;
- b) the functions $x = x(s)$, $y = y(s)$, defining Γ , belong to the space $C^{m+2,\alpha}$ (s is the arc length on Γ , $m \geq 0$, $0 < \alpha < 1$);

- c) if S is the length of Γ , then for $s = 0$ and $s = S$ the values of the functions $x(s)$ and $y(s)$ coincide together with their derivatives up to order $m + 2$, inclusive.

By $C^{m+2,\alpha}(\Gamma)$ we denote the totality of functions $\psi(s)$ for which the $(m + 2)$ -nd derivative satisfies the Hölder condition with exponent α , and for which the relations

$$\psi(0) = \psi(S), \quad \psi'(0) = \psi'(S), \dots, \psi^{m+2}(0) = \psi^{m+2}(S)$$

hold. The norm in this space is introduced in the usual way.

To each function $\psi(s) \in C^{m+2,\alpha}(\Gamma)$ in the space x, y, z there corresponds a closed curve $L_\psi : \{x = x(s), y = y(s), z = \psi(s)\}$.

Let P be an arbitrary point of the curve L_ψ . Denote by Q_p^+ and Q_p^- , respectively, the totality of planes T that pass through the tangent to L_ψ at the point P and leave the curve L_ψ above or below themselves.

Let $z = ax + by + c$ be the equation of the plane T . We shall call, respectively, the lower and upper twisting of the curve L_ψ the numbers

$$M(L_\psi) = \sup_{P \in L_\psi} \inf_{T \in Q_p^+} (a^2 + b^2), \quad M(L_\psi) = \sup_{P \in L_\psi} \inf_{T \in Q_p^-} (a^2 + b^2).$$

The numbers $M_n(L_\psi)$ and $M_v(L_\psi)$ are estimated from above as functions only of $\|\psi\|_{2,\beta}$, $\|x(s)\|_{2,\beta}$, $\|y(s)\|_{2,\beta}$, where β is any number in $(0, 1)$, and of the constant \varkappa_0 . These estimates were carried out by S. N. Bernstein ⁽⁴⁾ and are based on simple considerations from differential geometry.

Next, let $R(p^2 + q^2)$ be a positive continuous function on the p, q -plane. With the aid of the quantities $M_n(L_\psi)$ and $M_v(L_\psi)$ we construct, from this function, two new functions $F_n(p^2 + q^2; L_\psi; R)$ and $F_v(p^2 + q^2; L_\psi; R)$, taking them equal to the exact lower bound of the values of the function $R(p^2 + q^2)$ in the disk with center at the point (p, q) and with radius respectively $M_n(L_\psi)$ and $M_v(L_\psi)$. As is already reflected in the notation, the functions F_n and F_v depend only on the expression $p^2 + q^2$. It is obvious that for any p and q

$$F_n(p^2 + q^2; L_\psi; R) \leq R(p^2 + q^2); \quad F_v(p^2 + q^2; L_\psi; R) \leq R(p^2 + q^2).$$

The equality sign in these relations always occurs if

$$M_n(L_\psi) = M_v(L_\psi) = 0.$$

3. In this section we consider the first boundary-value problem for the equation

$$rt - s^2 = \varphi(x, y, z, p, q) \quad (2)$$

in the domain Ω , bounded by the closed convex curve Γ , under the boundary condition

$$z|_{\Gamma} = \psi(s). \quad (3)$$

Theorem 1. *Suppose the following conditions are satisfied:*

- 1) *The curve Γ satisfies requirements a) and b) of Section 2 and the number $m \geq 2$.*
- 2) *The function $\psi(s) \in C^{m+2, \alpha}(\Gamma)$.*
- 3) *For all finite z, p, q and any $(x, y) \in \Omega + \Gamma$ we have:*
 - a) $\varphi(x, y, z, p, q) \in C^{m, \alpha}(\Omega + \Gamma)$;
 - b) $\varphi(x, y, z, p, q) > 0$;
 - c) $\varphi_z(x, y, z, p, q) \geq 0$;
 - d) $\varphi(x, y, z, p, q) \leq \Phi_0 l(p^2 + q^2)$,

if $z \leq \max \psi(s)$, where $\Phi_0 = \text{const} > 0$ and $l(p^2 + q^2)$ is a continuous positive function on the p, q -plane. (The constant Φ_0 and the function $l(p^2 + q^2)$, generally speaking, depend on the function $\psi(s)$ and may change together with a change in $\max \psi(s)$.)

Then, if the inequality

$$\Phi_0 < \frac{\kappa_0^2}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_n \left(p^2 + q^2; L_\psi; \frac{1}{l} \right) dp dq, \quad (4)$$

is satisfied, the Dirichlet problem (2)–(3) always has a solution in $C^{m+2, \alpha'}(\Omega + \Gamma)$, and this solution is unique in the class of functions convex downward ($\alpha' \in (0, 1)$ is a certain number, determined by the data of the problem).

An analogous theorem holds in the class of convex functions which are convex upward, if $\varphi_z \leq 0$. In this case the solvability condition is given by the relation

$$\Phi_0 < \frac{\kappa_0^2}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_v \left(p^2 + q^2; L_\psi; \frac{1}{l} \right) dp dq.$$

If the function $\varphi(x, y, z, p, q) \equiv \Phi_0(1 + p^2 + q^2)^2$ ($\Phi_0 \equiv \text{const} > 0$), $\psi(s) \equiv 0$, and Γ is the circle $x^2 + y^2 = R^2$, then the solution of problem (2)–(3) will be the function

$$z = \sqrt{\frac{1}{\Phi_0} - R^2} - \sqrt{\frac{1}{\Phi_0} - x^2 - y^2}.$$

Therefore, in the particular case under consideration, a necessary condition for the solvability of problem (2)–(3) will be the inequality $\Phi_0 < \frac{1}{R^2}$. Since $\kappa_0 = \frac{1}{R}$, and

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_n \left(p^2 + q^2; L_0; \frac{1}{(1 + p^2 + q^2)^2} \right) dp dq = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp dq}{(1 + p^2 + q^2)^2} = \pi,$$

relation (4) turns into the inequality $\Phi_0 > \frac{1}{R^2}$.

Thus, the sufficient solvability condition (4) in the specific case presented is necessary.

4. Let us now turn to the consideration of the question of when problem (2)–(3) has several, and even infinitely many, regular solutions. In the work of M. A. Krasnosel'skii and the author⁵, this question was studied for generalized solutions of problem (2)–(3). Since the method of investigation in the case under consideration is completely analogous, we shall restrict ourselves to examining the essence of the matter in the following example.

In the domain $\Omega + \Gamma$, satisfying the conditions of Theorem 1, consider the first boundary-value problem

$$\frac{rt - s^2}{l(p^2 + q^2)} = f(z), \quad z|_{\Gamma} = 0.$$

In this case

$$F_n \left(p^2 + q^2; L_0; \frac{1}{l} \right) \equiv \frac{1}{l(p^2 + q^2)}.$$

Suppose that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp dq}{l(p^2 + q^2)} = +\infty.$$

Let the functions f and $l \in C^{m,\alpha}$ be strictly positive for finite values of their arguments.

Then to every convex function $z(x, y) \in C^{m, \alpha}(\Omega + \Gamma)$, with its convexity directed downward, there corresponds a function $v(x, y) \in C^{m+2, \alpha}(\Omega, \Gamma)$ which is a solution of the problem

$$\frac{v_{xx}v_{yy} - v_{xy}^2}{l(v_x^2 + v_y^2)} = f(z(x, y)), \quad v|_{\Gamma} = 0.$$

The function $v(x, y)$ is convex and has its convexity directed downward.

Thus, on the cone of convex nonpositive functions in the space $C^{m, \alpha}(\Omega + \Gamma)$, there is defined a completely continuous operator $v = F(z)$ which maps this cone into itself.

Applying to this operator M. A. Krasnosel' skii' s theorems on fixed points in cones⁶, we readily establish, in the case when the function $f(z)$ has the form of a staircase with appropriately chosen heights of the steps and horizontal plateaus, that problem (2)–(3) has at least a countable number of regular solutions.

The considerations given in § 4 are easily generalized to the general case of the first boundary-value problem

$$rt - s^2 = \varphi(x, y, z, p, q), \quad z|_{\Gamma} = 0.$$

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Received
18 II 1964

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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