E. M. SEMENOV

1964

SovietRxiv

Abstract

Full Text

E. M. SEMENOV

EMBEDDING THEOREMS FOR BANACH SPACES OF MEASURABLE FUNCTIONS

(Presented by Academician A. N. Kolmogorov on 31 I 1964)

Recently, various authors (^{1,2}) have studied Banach spaces of measurable functions possessing the following property: if $x(t) \in E$, then $|x(t)| \in E$, and the norms of these functions in E coincide.

In the present note we consider spaces of measurable functions E satisfying the requirements:

1) If $x(t) \in E$ and the functions $|x(t)|$ and $|y(t)|$ are equimeasurable, then $y(t) \in E$ and
$$\|x\|_E = \|y\|_E.$$
2) If $|x(t)| \leq |y(t)|$ and $y(t) \in E$, then $x(t) \in E$ and
$$\|x\|_E \leq \|y\|_E.$$

We shall call such spaces symmetric. We shall assume that the functions from E are defined on the interval $[0,1]$.

It is obvious that $\chi_e(t)$, the characteristic function of any measurable subset $e \subset [0,1]$, belongs to E, whence it follows that every finitely valued function belongs to E. It can be shown (see (^{3}) or (^{4})) that the second condition in the definition of a symmetric space follows from the first if it is known that the set of finitely valued functions is dense in E.

Among symmetric spaces, the spaces $\Lambda(\varphi)$ and $M(\varphi)$, first considered by Lorentz (^{5}) and Halperin (^{6}), possess important extremal properties. By $\Lambda(\varphi)$ is denoted the Banach space of functions measurable on $[0,1]$ for which

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(t)\, d\varphi(t),$$

where $x^*(t)$ is a nonincreasing function equimeasurable with $|x(t)|$, and $\varphi(t)$ is a nondecreasing concave function on $[0,1]$. In $M(\varphi)$ the norm is introduced by the formula

$$\|x\|_{M(\varphi)} = \sup_{0<h\leq 1} \frac{\int_0^h x^*(t)\, dt}{\varphi(h)}.$$

Here $\varphi(t)$ does not decrease on $[0,1]$, and $\varphi(t) > 0$ for $t > 0$.

By definition, $\|\chi_e\|_E$ is determined by specifying the measure of the set $e \subset [0,1]$. In other words, for every symmetric space E there exists a function $\varphi(t)$ $(0 \le t \le 1)$ such that

$$\|\chi_e\|_E = \varphi(me).$$

From the definition of a symmetric space it follows that $\varphi(0) = 0$. In this case we shall call the function $\varphi(t)$ fundamental for the space E. For example, $\varphi(t) = \sqrt[p]{t}$ is the fundamental function of the space $\mathcal{L}_p$ $(1 \le p < \infty)$.

Theorem 1. *In order that a function $\varphi(t)$ be fundamental for some symmetric space, it is necessary and sufficient that the following conditions be satisfied: a) $\varphi(t)$ does not decrease on $[0,1]$; b) $\varphi(t)/t$ does not increase on $(0,1]$.*

Proof. Necessity. Let $0 < t_1 < t_2 \le 1$, $e_1 \subset e_2 \subset [0,1]$, $me_1 = t_1$, $me_2 = t_2$, and $\|\chi_e\|_E = \varphi(me)$. Then

$$\varphi(t_1) = \|\chi_{e_1}\|_E = \frac{1}{2}\|(\chi_{e_1} - \chi_{e_2/e_1}) + \chi_{e_2}\|_E \le \frac{1}{2}\|\chi_{e_1} - \chi_{e_2/e_1}\|_E + \frac{1}{2}\|\chi_{e_2}\|_E = \|\chi_{e_2}\|_E = \varphi(t_2).$$

Put

$$e_{k,n,m}(t) = \chi_{[0,\frac{k}{n}]}(t) - \chi_{[\frac{m-1}{n},\frac{m}{n}]}(t),$$

where $1 \le m \le k \le n$. From the obvious identity

$$(k-1)\chi_{[0,\frac{k}{n}]}(t) = \sum_{m=1}^{k} e_{k,n,m}(t)$$

we have, by definition,

$$(k-1)\left\|\chi_{[0,\frac{k}{n}]}\right\|_E = \left\|\sum_{m=1}^{k} e_{k,n,m}\right\|_E \le \sum_{m=1}^{k} \|e_{k,n,m}\|_E = k\left\|\chi_{[0,\frac{k-1}{n}]}\right\|_E$$

or

$$(k-1)\varphi\left(\frac{k}{n}\right) \le k\varphi\left(\frac{k-1}{n}\right),$$

whence we obtain

$$\frac{\varphi\left(\frac{k}{n}\right)}{\frac{k}{n}} \le \frac{\varphi\left(\frac{k-1}{n}\right)}{\frac{k-1}{n}}.$$

Thus, if $r < R$ are rational, then

$$\frac{\varphi(R)}{R} \le \frac{\varphi(r)}{r}. \tag{1}$$

Now let $0 < t_1 < t_2 \le 1$. Construct two sequences of rational numbers r_k and R_k such that $r_k \uparrow t_1$, $R_k \downarrow t_2$. Then, by virtue of a) and (1):

$$\frac{\varphi(t_2)}{t_2} \le \frac{R_k}{t_2}\frac{\varphi(R_k)}{R_k} \le \frac{R_k}{t_2}\frac{\varphi(r_k)}{r_k} \le \frac{R_k t_1}{r_k t_2}\frac{\varphi(t_1)}{t_1}. \tag{2}$$

Passing in (2) to the limit as $k \to \infty$, we obtain the required inequality.

Sufficiency. Let $\varphi(t)$ satisfy conditions a) and b). We shall prove that as E one may take the space $M(\psi)$, $\psi(t) = t/\varphi(t)$. Indeed,

$$\|\chi_e\|_{M(\psi)} = \sup_{0<h\le 1} \frac{\int_0^h \chi_e^*(t)\,dt}{\psi(h)}.$$

Since $\psi(h)$ is nondecreasing in h, it follows that

$$\|\chi_e\|_{M(\psi)} = \sup_{0<h\le me} \frac{\int_0^h dt}{\psi(h)} = \sup_{0<h\le me} \varphi(h).$$

But $\varphi(h)$ also is nondecreasing in h, therefore

$$\|\chi_e\|_{M(\psi)} = \varphi(me).$$

The theorem is proved.

Theorem 2. *If E is symmetric, then $\mathcal{L}_\infty \subset E \subset \mathcal{L}_1$, and both embeddings are continuous.*

Proof. If $x(t) \in \mathcal{L}_\infty$, then, by the definition of a symmetric space, $x(t) \in E$ and

$$\|x\|_E \le \|\operatorname{vrai\,sup}|x(t)|\|_E = \varphi(1)\|x\|_{\mathcal{L}_\infty}.$$

Let

$$y_0(t) = \sum_{k=0}^{N-1} x_k \chi_{e_k}(t), \quad \text{where } x_k \ge 0, \quad me_k = \frac{1}{N}.$$

Put

$$y_j(t) = \sum_{k=0}^{N-1} x_{k+j(\bmod N)} \chi_{e_k}(t)$$

$(j = 1, 2, \dots, N-1)$. Obviously, the functions $y_0(t)$ and $y_j(t)$ are equimeasurable, and therefore $\|y_0\|_E = \|y_j\|_E$. Since

$$\sum_{k=0}^{N-1} x_k = \sum_{j=0}^{N-1} y_j(t),$$

we have

$$\sum_{k=0}^{N-1} x_k\varphi(1) = \left\| \sum_{j=0}^{N-1} y_j \right\|_E \leq \sum_{j=0}^{N-1} \|y_j\|_E = N\|y_0\|_E,$$

whence

$$\|y_0\|_E \geq \frac{1}{N} \sum_{k=0}^{N-1} x_k\varphi(1) = \varphi(1)\|y_0\|_{L_1}. \tag{3}$$

Now let $z(t)$ be a nonnegative function from E. There exists a nondecreasing sequence of nonnegative simple functions $z_k(t)$ converging to $z(t)$ almost everywhere on $[0,1]$. Moreover, one may assume that the function $z_k(t)$ is constant on sets of equal measure. Since $z_k(t) \leq z(t)$, we have $\lim_{k\to\infty} \|z_k\|_E \leq \|z\|_E$. By Levi' s theorem it follows that $\lim_{k\to\infty} \|z_k\|_{L_1} = \|z\|_{L_1}$ (if $z(t) \notin \mathcal{L}_1$, then, obviously, $\lim_{k\to\infty} \|z_k\|_{L_1} = \infty$). From (3) it follows that

$$\|z\|_{L_1} = \lim_{k\to\infty} \|z_k\|_{L_1} \leq \varphi(1) \lim_{k\to\infty} \|z_k\|_E \leq \varphi(1)\|z\|_E. \tag{4}$$

Replacing an arbitrary function from E by its modulus and using (4), we obtain the required inequality. The theorem is proved.

It follows from Theorem 1 that the fundamental function $\varphi(t)$ is continuous on $(0,1]$. It can be shown (see ([3]) or ([4])) that for the continuity of $\varphi(t)$ at $t = 0$ it is necessary and sufficient that the norm of the space E be weaker than the norm of $\mathcal{L}_\infty$; similarly, the boundedness on $(0,1]$ of the function $\varphi(t)/t$ is equivalent to the fact that the norm of E is stronger than the norm of $\mathcal{L}_1$.

Theorem 3. Let $\varphi(t)$ be the fundamental function of the symmetric space E. Then there exists a concave function $\psi(t)$ satisfying the conditions $\psi(0) = \varphi(0)$, $\psi(1) = \varphi(1)$, $\varphi(t) \leq \psi(t) \leq 2\varphi(t)$, and such that $E \supset \Lambda(\psi)$ and

$$\|x\|_E \leq \|x\|_{\Lambda(\psi)}.$$

If $\varphi(t)$ is concave, then $\psi(t) = \varphi(t)$.

In the proof of the theorem the following simple fact is used: the functions

$$\frac{1}{\psi(me \cup g)}[\chi_e(t) - \chi_g(t)],$$

where $e, g \subset [0,1]$, $e \cap g = \emptyset$, and only they, are the extreme points of the unit ball of the space $\Lambda(\psi)$.

In what follows (in the corollaries of Theorems 3 and 7) we shall use the definition from ([7],[8]).

Corollary 1. If $\varphi_0(t) \le \varphi_1(t)$, then the spaces $\Lambda(\varphi_1^{-\alpha}\varphi_0^{\alpha})$ $(0 \le \alpha \le 1)$ form a maximal continuous normal scale of spaces connecting $\Lambda(\varphi_0)$ and $\Lambda(\varphi_1)$.

Corollary 2 (interpolation theorem in the spaces $\Lambda(\varphi)$). If $\varphi_0(t) \le \varphi_1(t)$, $\psi_0(t) \le \psi_1(t)$, A is a linear operator and

$$\|Ax\|_{\Lambda(\varphi_i)} \le C_i \|x\|_{\Lambda(\psi_i)} \quad (i = 0, 1),$$

then for all $\alpha \in (0,1)$ and all $x \in \Lambda(\psi_0^{1-\alpha}\psi_1^{\alpha})$ the inequality

$$\|Ax\|_{\Lambda(\varphi_0^{1-\alpha}\varphi_1^{\alpha})} \le C_0^{1-\alpha} C_1^{\alpha} \|x\|_{\Lambda(\psi_0^{1-\alpha}\psi_1^{\alpha})}$$

is satisfied.

The proof follows directly from Corollary 1 and the interpolation theorem of S. G. Kreĭn (^{7}).

Theorem 4. If $\varphi(t)$ is the fundamental function of a symmetric space E, then $E \subset M(\psi)$ and

$$\|x\|_{M(\psi)} \le \|x\|_E,$$

where $\psi(t) = t/\varphi(t)$.

Theorem 4 cannot be improved in the sense that $\|\chi_e\|_{M(\psi)} = \varphi(me)$; the same can be said also of Theorem 3, if $\varphi(t)$ is concave.

From Theorems 3 and 4 one can obtain new embedding theorems for the basic classes of symmetric spaces: $\mathcal{L}_M$(^{9}), $\Lambda(p,\varphi)$, and $M(p,\varphi)$ (^{5}).

Theorem 5. Let E_i $(i = 0, 1)$ be symmetric spaces and let $\varphi_i(t)$ be their corresponding fundamental functions. If $\varphi_0(t)$ and $t/\varphi_1(t)$ are concave on $[0,1]$ and

$$C = \int_0^1 \varphi_0'(t)\, d\,\frac{t}{\varphi_1(t)} < \infty,$$

then $E_0 \supset E_1$ and

$$\|x\|_{E_0} \le C\|x\|_{E_1}.$$

Theorem 6. If the function $\psi(t) = t x_0^*(t)$ is nondecreasing in some neighborhood of 0, $x^*(1) > 0$, and

$$\varlimsup_{h\to 0}\frac{\displaystyle\int_0^h x_0^*(t)\,dt}{hx_0^*(h)} < \infty,$$

then from the inclusion $x_0(t) \in E$, where E is symmetric, it follows that $x_0(t) \in M(\psi) \subset E$, and the embedding is continuous.

Let E be a symmetric space. A set of functions $M \subset E$ will be called symmetric if $x(t) \in E$ whenever $y(t) \in E$ and $x^*(t) \leq y^*(t)$. It is obvious that the unit ball of the space E is a symmetric set. We give one property of embeddings of symmetric spaces.

Theorem 7. Let the set of finitely valued functions be dense in the symmetric spaces E_0 and E_1, let $\|x\|_{E_0} \leq \|x\|_{E_1}$, and let M be a symmetric set from E_1. If M is closed in the norm of E_1, then it is also closed in the norm of E_0.

Applying the theorem to the unit ball of the space E_1, we obtain, by virtue of ([8]),

Corollary. If for the spaces E_0 and E_1 the conditions of Theorem 7 are fulfilled, then E_0 and E_1 are conjugate.

The author expresses sincere gratitude to S. G. Kreĭn for guidance.

Voronezh State University

Received
31 I 1964

CITED LITERATURE

[1] W. A. J. Luxemburg, A. C. Zaanen, *Math. Ann.*, **14**, 150 (1963).
[2] Yu. I. Gribanov, P. K. Belobrov, *Izv. vyssh. uchebn. zaved., Matematika*, No. 4, 44 (1963).
[3] E. M. Semenov, *Scales of Banach spaces connecting the spaces $\mathcal{L}_1$ and $\mathcal{L}_\infty$*, Dissertation, Voronezh, 1964.
[4] B. S. Mityagin, A. S. Shvarts, *UMN*, **19** (2), No. 2 (1964).
[5] G. G. Lorentz, *Pacific J. Math.*, **1**, 411 (1950).
[6] J. Halperin, *Canad. J. Math.*, **5**, 273 (1953).
[7] S. G. Kreĭn, *DAN*, **132**, No. 3 (1960).
[8] S. G. Kreĭn, Yu. I. Petunin, *DAN*, **139**, No. 6 (1961).
[9] M. A. Krasnosel' skii, Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, 1958.

Note: Figure translations are in progress. See original paper for figures.