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Abstract

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REGULARITY, PRODUCT, AND SPECTRA OF PROXIMITY SPACES

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Here one important property of δ-spaces* is studied—the property of possessing the greatest uniform structure, which we have called **regularity**; a new definition of the Cartesian product of δ-spaces is introduced, natural from many points of view, and in connection with this several ways of introducing proximity into the inverse set of the inverse spectrum of a δ-space are investigated.

Yu. M. Smirnov ([6]) showed that every metrizable and every bicompact δ-space is regular, and posed the question of the regularity of an arbitrary δ-space. The answer turned out to be negative. Here several necessary and sufficient conditions for regularity are given. To each δ-space P, by changing in the minimally possible way the number of pairs of close sets, we associate a regular δ-space $P!$—a correction—and study their mutual properties. In doing so, the apparatus of the product of δ-spaces is used. As is known, for topological and for uniform spaces the Cartesian product has a simple and, most importantly, uniquely natural (apart from equivalent ones) definition. Giving the first definition (see ([3])) of the Cartesian product of δ-spaces—we shall call it **weak**—V. A. Efremovich already pointed out its defect: even the product of two lines is not δ-homeomorphic to the plane—it is not even metrizable! The strong product introduced here is free of this and analogous defects. The results obtained make the author think that these two definitions are the basic ones. A similar situation is also found for inverse spectra of δ-spaces: of the seven naturally possible definitions only two turn out to be distinct.

The last three theorems give spectral characteristics of complete, strongly complete, and, finally, complete regular δ-spaces.

Definition 1. Let $\{P_\alpha\}$ be a system of δ-spaces; we shall call sets A and B of the abstract product ΠP_α **separable** if there exists a Cartesian product** of a finite number of uniform*** coverings no element of which intersects both A and B simultaneously; the **strong product** $\check{\Pi} P_\alpha$ (or $P \times Q$) of δ-spaces P_α is the set ΠP_α, in which the sets A and B are considered distant if and only if there exist finite systems $\{A_i\}$ and $\{B_j\}$ such that $A = \bigcup A_i$, $B = \bigcup B_j$, and for all i and j, A_i and B_j are separable. All the axioms of δ-spaces turn out to be satisfied.

Theorem 1. *The strong product $\check{\Pi} P_\alpha$ of δ-spaces P_α is the smallest of all δ-spaces of the abstract product ΠP_α in which the Cartesian product of any finite* *system of uniform coverings ω_{α_i} is uniform.*

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* For brevity, we shall henceforth call proximity spaces δ-spaces, and proximity-continuous mappings δ-continuous (see ([2]) or ([6]), p. 543).

** By the Cartesian product of coverings ω_{α_i} of the corresponding sets P_{α_i} we shall mean the covering of the product ΠP_α consisting of all its subsets of the form ΠO_α, where $O_\alpha \in \omega_{\alpha_i}$ if $\alpha = \alpha_i$, and $O_\alpha = P_\alpha$ otherwise.

*** The definition of a uniform covering of a proximity space was given by Yu. M. Smirnov in ([6]), p. 568.

Corollary 1. The strong product $\overset{\times}{\prod} P_\alpha$ is the upper bound of the family of δ-spaces generated on $\prod X_\alpha$ by all such products $\prod X_\alpha$ of uniform spaces X_α, each of which generates the corresponding δ-space P_α.

Corollary 2. If each space P_α is proper (in particular, metrizable), then $\overset{\times}{\prod} P_\alpha$ is generated by the product $\prod \overset{\times}{P}_\alpha$ of maximal uniform structures $\overset{\times}{P}_\alpha$, generating the corresponding δ-spaces P_α.*

Corollary 3. The strong product of no more than a countable number of metrizable δ-spaces is metrizable.

Corollary 4. $\dot{\prod} P_\alpha < \overset{\times}{\prod} P_\alpha$.

Theorem 2. The products $P \cdot Q$ and $P \times Q$ canonically coincide if and only if at least one of the factors is totally bounded.**

Theorem 3. Let X and Y be equimorphic*** uniform spaces, and let Z be a precompact space; then the products $X \times Z$ and $Y \times Z$ are canonically equimorphic.

Theorem 4. The strong (and weak) product of a finite number of totally bounded δ-spaces is totally bounded.

Definition 2. A δ-space will be called **strongly complete** if it is generated by at least one complete uniform space. Every strongly complete δ-space is complete; every proper complete δ-space is strongly complete.

Lemma. A closed subset of a strongly complete δ-space is strongly complete.****

Theorem 5. The weak (strong) product of complete (strongly complete) δ-spaces is complete (strongly complete).

Theorem 6. The following properties of a δ-space P are equivalent:

A. P is proper.

B. The projection of the diagonal of the product $P \times P$ onto P is an equimorphism.

C_1, C_2, C_3. For any δ-continuous mappings $f : P \to X$ and $g : P \to Y$, the combination $(f, g) : P \to X \times Y$ is δ-continuous; $X \times Y$ is the strong product of proximity spaces (respectively, the product***** of metric spaces, the product of uniform spaces).

D. The sum (maximum) of any two δ-continuous pseudometrics****** of the space P is δ-continuous.

Corollary 1. The proximity of a completely regular space generated by the Čech extension is proper.

Corollary 2. If Q is dense in P and is proper, then P is proper.

Let us note that the property of properness is not inherited by closed sets even in totally bounded spaces.

Definition 3. By the **correction** $P!$ of a proximity space P we shall mean the weakest proper space defined on the set P and having a proximity stronger than P.

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* For comparison, we point out that the weak product $\dot{\prod} P_\alpha$ (or $P \cdot Q$) is generated by the product $\dot{\prod} P_\alpha$ of minimal uniform structures $\dot{P}_\alpha$, generating the corresponding δ-spaces P_α.

** A δ-space is called **totally bounded** (see (7), p. 291) if its completion is bicompact.

*** That is, proximally homeomorphic.

**** For complete spaces, a proposition analogous to this was proved by Yu. M. Smirnov ((7), p. 285).

***** That is, the space with metric $\rho = \sqrt{\rho_1^2 + \rho_2^2}$.

****** It is not enough here to restrict oneself only to Euclidean spaces $X = E^n$ and $Y = E^k$.

******* A pseudometric ρ of a δ-space P is called δ-continuous if, for every pair of close sets A and B, $\rho(A, B) = 0$ ((7), p. 295).

Theorem 7. *Every proximity space P has a completion; the topologies of P and $P!$ coincide.*

Denote by ΔP the δ-space obtained by the natural identification of its points with the points of the diagonal of $P \times P$. For what follows, define by transfinite induction the δ-spaces $\Delta^\chi P$: $\Delta^{\chi+1} P = \Delta\Delta^\chi P$ and $\Delta^\chi P = \sup\{\Delta^\alpha P \mid \alpha < \chi\}$* for limit indices χ. Beginning with some ν, the sequence $\Delta^\chi P$ becomes stationary: $P! = \Delta^\nu P$.

Theorem 8. *If $f : P \to Q$ is δ-continuous, then $f : P! \to Q!$ is δ-continuous.*

Theorem 9. $(\dot{\Pi} P_\alpha)! = (\dot{\Pi} P_\alpha)! = (\dot{\Pi} P_\alpha!)!$

Corollary 1. *The product $P \cdot Q$ is improper if both P and Q are not totally bounded.*

Corollary 2. *If the proximities of uniform spaces X and Y are not totally bounded, and one of them is precompact, then the proximity in the product $X \times Y$ is improper**.*

Corollary 3. *The weak product of metric spaces $R \dot{\times} S$ is nonmetrizable, unless it coincides with the strong product $R \times S$.*

Theorem 10. *A proper δ-space is a completion only of itself if and only if it is totally bounded.*

Corollary. *A completely regular (normal) space is generated only by proper δ-spaces if and only if it is pseudocompact (countably compact).*

Theorem 11. *The product $P \times Q$ is proper if P contains a dense metrizable subspace, and Q is a generalized sequential bicompactum (i.e. in each of its subsets, for every point of contact, there is a subset of regular cardinality converging in cardinality to this point).*

The question of the propriety of the strong product of proper δ-spaces is connected with the problem of the coincidence of δ-dimensions***.

Theorem 12. *The product $N \times \beta N$ (N is a countable discrete space) is proper if $\Delta d(N \times \beta N) = 0$ and only in that case**.*

Let $\Sigma = \{P_\alpha; \pi_\alpha^\beta\}$ be an inverse spectrum of δ-spaces with δ-continuous projections. We shall turn the set Π of all its "threads" into a δ-space in the following ways:

Definitions of $\dot{\Sigma}$ and $\overset{\times}{\Sigma}$. By the **limit** $\lim \Sigma$ (respectively, the **strong limit** $\lim \overset{\times}{\Sigma}$) of the spectrum Σ we shall mean the set Π, endowed with the proximity of the product $\dot{\Pi} P_\alpha$ (of the product $\overset{\times}{\Pi} P_\alpha$), in which it is naturally contained.

Theorem 13. *Of the following definitions $\Sigma^F, \Sigma', \Sigma_f^F, \Sigma'_f, \Sigma_s$ of proximity on the set Π, the first is equivalent to the definition $\overset{\times}{\Sigma}$, and the others—to the definition $\dot{\Sigma}$; always $\lim \Sigma < \lim \overset{\times}{\Sigma}$; if the spaces P_α are proper, then $\dot{\Sigma}$ is equivalent to $\overset{\times}{\Sigma}$.*

Σ^F. Fix a finite set F of indices α, and in each space P_α (for $\alpha \in F$) choose a uniform cover ω_α; further, in each cover ω_α mark one element U_α and consider the subset U of the set Π, consisting of all such points (p_α) that $\alpha \in F$ entails—

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* If there is an increasing sequence Z_α of δ-spaces, then their supremum is the proximity in which two sets are distant if they are distant in some Z_α, and only then.

** This is a generalization of the first examples of improper δ-spaces of Katětov ((5), 4.3 and 2.3) and Dowker ((1), p. 139).

*** The small dimension δd was defined by Yu. M. Smirnov (9) using finite uniform covers, and the large dimension Δd—by J. R. Isbell (4) using arbitrary uniform covers. Always $\delta dP < \Delta dP$. The question of the coincidence of these dimensions for δ-spaces remains open.

**** Obviously, $\delta d(N \times \beta N) = 0$.

there exists $p_\alpha \in U_\alpha$. The system of all sets of this kind is, obviously, a cover of the set Π. We shall call the sets A and B **separable** if at least one such cover contains no element intersecting both A and B at the same time, and **distant** if they are unions of a finite number of pairwise separable sets.

Σ'. This definition is obtained from the preceding one by replacing the first six words by the following: "fix one index β."

Σ_f^F and Σ_f'. These definitions are obtained from the preceding two if, instead of arbitrary uniform covers, one takes only finite uniform covers.

Σ_s. Sets A and B lying in Π are considered distant if, for some index α, the traces of the sets A and B in the space P_α are distant.

Consequence. The limit of a countable spectrum of metrizable δ-spaces is metrizable.

Theorem 14. The limit (strong limit) of a spectrum of complete (strongly complete) δ-spaces is complete (strongly complete).

Definition 4. An inverse spectrum $\{P_\alpha; \pi_\alpha^\beta\}$ will be called **uniform** if, for every uniform cover ω of its limit P, some projection $\pi_\alpha : P \to P_\alpha$ is an ω-mapping*.

Theorem 15. Let $\{\Sigma_\lambda\} = \{\{\,^\lambda P_\alpha;\,^\lambda\pi_\alpha^\beta \mid \alpha, \beta \in A\} \mid \lambda\}$ be a system of inverse spectra; then for the spectra

$$\Sigma = \{\dot{\Pi}^\lambda P_\alpha;\, \Pi^\lambda \pi_\alpha^\beta\} \quad \text{and} \quad T = \{\check{\Pi}^\lambda P_\alpha;\, \Pi^\lambda \pi_\alpha^\beta\}$$

** we have

$$\lim \Sigma = \Pi \lim \Sigma_\lambda \quad \text{and} \quad \dot{\Pi} \lim \Sigma_\lambda < \lim T < \check{\Pi} \lim \Sigma_\lambda;$$

if all spectra Σ_λ are uniform, then

$$\lim T = \check{\Pi} \lim \Sigma_\lambda.$$

Theorem 16. The limit of a uniform spectrum is regular if the δ-spaces forming it are regular.

Definition 5. By a **correction of an inverse spectrum**

$$\Sigma = \{P_\alpha; \pi_\alpha^\beta\}$$

we shall call the spectrum

$$\Sigma! = \{P_\alpha!; \pi_\alpha^\beta\}.$$

Theorem 17.

$$(\lim \Sigma)! = (\lim \Sigma!) = (\lim \Sigma!)!$$

Theorem 18. The class of complete δ-spaces coincides with the class of limits of (uniform) spectra of complete δ-spaces with metrizable corrections.

Theorem 19. The class of strongly complete δ-spaces coincides with the class of limits of spectra of complete metrizable δ-spaces.

Theorem 20. The class of complete regular δ-spaces coincides with the class of limits of uniform spectra of complete metrizable δ-spaces.

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* That is, the inverse image of some uniform cover ω_α of the space P_α is inscribed in ω.

** The mapping $\Pi\pi_\lambda : \Pi X_\lambda \to \Pi Y_\lambda$ assigns to each point (X_λ) the point $(\pi_\lambda X_\lambda)$.

Note: Figure translations are in progress. See original paper for figures.