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Abstract

Full Text

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ASYMPTOTIC FORMULAS FOR THE EIGENVALUES OF THE LAPLACE OPERATOR IN THE CASE OF A POLYHEDRON

(Presented by Academician A. A. Dorodnitsyn, 3 III 1964)

1. In the present paper we study the asymptotics of the eigenvalues of the boundary-value problem for the equation

$$-\Delta u = k^2 u, \tag{1}$$

considered in an m-dimensional polyhedron Π, under one of the two boundary conditions:

$$u\big|_{\Gamma} = 0 \tag{2}$$

or

$$\frac{\partial u}{\partial n}\bigg|_{\Gamma} = 0. \tag{2'}$$

It is known (see, for example, (^{1})) that in the case of an arbitrary finite domain D with piecewise smooth boundary the asymptotic formula

$$n(k) = \frac{\operatorname{mes} D}{2^m \pi^{m/2} \Gamma(m/2+1)}\, k^m + O(k^{m-1} \ln k) \tag{3}$$

is valid as $k \to \infty$. Here $n(k)$ is the number of eigenvalues of the problem not exceeding k^2.

In this paper we show that in the case of a polyhedron the remainder in formula (3) admits a more precise estimate, namely $O(k^{m-1})$. In addition, for functions that are obtained by successive integration of $n(k)$, we find the following terms of the asymptotics.

Let us formulate the main result.

Theorem. Let the boundary-value problem (1), (2) or (1), (2′) be given on an m-dimensional polyhedron. Then, for $0 \le p \le m-1$ and as $k \to \infty$,

$$\frac{1}{\Gamma(p+1)}\int_0^k (k-t)^p\,dn(t)=\sum_{l=1}^{m} a_l \frac{\Gamma(l+1)}{\Gamma(p+l+1)}\,k^{p+l}+O(k^{m-1}). \tag{4}$$

The first three coefficients in formula (4) are expressed explicitly in terms of the geometric characteristics of the polyhedron, namely:

$$a_m=\frac{\operatorname{mes}_{(m)}\Pi}{2^m\pi^{m/2}\Gamma(m/2+1)},\qquad a_{m-1}=\mp\frac{\sum_i \operatorname{mes}_{(m-1)} S_i^{m-1}}{2^{m+1}\pi^{(m-1)/2}\Gamma((m+1)/2)},$$

$$a_{m-2}=\frac{1}{2^{m+1}\pi^{(m-2)/2}\Gamma(m/2)}\sum_i \frac{1}{3}\left(\frac{\omega_i}{\pi}-\frac{\pi}{\omega_i}\right)\operatorname{mes}_{(m-2)} S_i^{m-2},$$

where S_i^k is a k-dimensional face of the polyhedron, and ω_i is the magnitude of the dihedral angle at the face S_i^{m-2}. The minus sign corresponds to problem (1), (2), and the plus sign to problem (1), (2′).

It was not possible to obtain an explicit expression for the remaining coefficients. They can be expressed with the help of the Green function of the mixed problem for the wave equation on a spherical polyhedron.

2. Formula (4) is a generalization to the m-dimensional case of the result obtained by us earlier for the case of a plane polygon (see $(^2)$) and is proved by analogous methods*. We consider the generalized function $w(t)$ on the space K of finite infinitely differentiable functions, equal to

$$w(t)=\sum_{n=1}^{\infty}\cos k_n t=\int_0^{\infty}\cos kt\,dn(k), \tag{5}$$

where k_n^2 is the n-th eigenvalue of problem (1), (2), (1), (2′).

Essential for what follows is the following

Lemma. There exists a $t_0>0$ such that for $|t|<t_0$ the formula

$$w(t)=\sum_{i=-m}^{0} b_i \frac{|t|^i}{\Gamma((i+1)/2)} \tag{6}$$

holds.

Here $\frac{|t|^i}{\Gamma((i+1)/2)}$ are even homogeneous generalized functions, and b_i are certain constants determined by the geometry of the polyhedron.

Using (6) and applying Tauberian theorems for Fourier integrals to $w(t)$, we obtain formula (4).

Let us outline the proof of the lemma. Consider the mixed problem for the wave equation

$$\Delta w(x,y,t)=\frac{\partial^2}{\partial t^2}w(x,y,t),$$

$$w(x,y,0)=\delta(|x-y|),\qquad w'_t(x,y,0)=0 \tag{7}$$

with boundary condition $w|_\Gamma=0$ or $\left.\frac{\partial w}{\partial n}\right|_\Gamma=0$, where x,y are points of the m-dimensional space and $|x-y|$ is the distance between them.

The solution of problem (7) will be called the Green' s function of the mixed problem for the wave equation. We consider the Green's function as a generalized function in t on the space K; x and y are regarded as parameters. It is not difficult to show that $w(t)$ is the trace of the Green' s function, i.e.

$$w(t)=\int_\Pi w(x,x,t)\,dx.$$

For sufficiently small t, $|t|<t_0$, the Green' s function $w(x,y,t)$, by virtue of the finiteness of the propagation speed of perturbations and the uniqueness of the solution of problem (7), coincides with the Green' s function for one of the polyhedral angles of the polyhedron or with the Green' s function of the whole space; the latter are homogeneous, as functions of three variables, of degree $-m$. Using this fact, one can represent $w(t)$ for $|t|<t_0$ in the form of a finite sum of homogeneous functions, namely:

$$w(t)=\sum_{i=0}^{m}w_i(t). \tag{8}$$

Here $w_i(t)$ are even homogeneous functions of degree $-i$. Since an even homogeneous function of one variable of degree λ is, up to a constant factor,

$$\frac{|t|^\lambda}{\Gamma((\lambda+1)/2)}$$

(see (4)), formula (8) gives the expression of the trace up to the coefficients b_i. The lemma is proved.

* After paper (2) had already been published, the author became acquainted with the work of Bailey and Brownell (3), where it is also proved that, in the case of a plane polygon, the logarithmic factor is absent in the remainder term of the asymptotics of $n(k)$. The method of Bailey and Brownell differs from ours.

To compute the coefficients b_i, one must know explicit expressions for the Green' s functions for polyhedral angles and for the whole space. Such expressions are known for the Green' s function of the whole space, of a half-space, and of a dihedral angle (see (²)). Thus, in the planar case $w(t)$ is computed explicitly. In the general case, the first three coefficients are computed.

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Note: Figure translations are in progress. See original paper for figures.