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Academician of the Academy of Sciences of the Azerbaijan SSR Z. I. KHALILOV

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Abstract

Full Text

MATHEMATICS

Academician of the Academy of Sciences of the Azerbaijan SSR Z. I. KHALILOV

A LINEAR CONTROL PROBLEM IN A BANACH SPACE

Let B be a real Banach space. A **cone** ^(1,2) in the space B is a closed convex set $K \subset B$ possessing the property that, if $x \in K$, $x \neq 0$, then $\lambda x \in K$ for $\lambda \geq 0$ and $\lambda x \notin K$ for $\lambda < 0$, and if $x, y \in K$, then $x + y \in K$. We introduce a partial order in the Banach space B with the cone K in the usual way: we write $x \leq y$ if $y - x \in K$.

A linear (continuous) functional X , defined on all of B , will be called **positive (nonnegative)** on the cone K if $(x, X) > 0$ (≥ 0) for all $x \in K$; by the symbol (x, X) we mean the value of the linear functional X at the point x .

If the difference $X_1 - X_2 > 0$, then we shall write $X_1 > X_2$. If X_1 and X_2 have the property: either $X_1 > X_2$, or $X_1 < X_2$, or $X_1 = X_2$, then we shall say that the linear functionals X_1 and X_2 are comparable in the cone K .

We shall need two generalizations of the Riemann-Stieltjes (R.-S.) integral to vector functions:

$$\int_a^b (f(t), dg(t)), \quad (1)$$

$$\int_a^b A(t) dg(t), \quad (2)$$

where $f(t)$ is a vector function with values in B ; $g(t)$ is a vector function with values in B^* (the conjugate space); $A(t)$ is a bounded linear operator acting, for each fixed t , in B . For this, divide the interval $[a, b]$ into n parts by the points $a = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n = b$ and arbitrarily fix the points τ_i ($t_{i-1} \leq \tau_i \leq t_i$), $i = 1, 2, \dots, n$. By the integral (1) we mean the limit of the sum

$$S_\lambda(f, g) = \sum_{i=1}^n (f(\tau_i), g(t_i) - g(t_{i-1})),$$

when $\lambda = \max |t_i - t_{i-1}| \rightarrow 0$; by the integral (2) we mean the limit, in a certain topology, of the sum

$$S_\lambda(A, g) = \sum_{i=1}^n A(\tau_i) [g(t_i) - g(t_{i-1})],$$

when $\lambda \rightarrow 0$ (in the strong or weak topology).

Other generalizations of the R.-S. integral are given in ⁽³⁾.

Many properties of the classical R.-S. integral are preserved for the integrals (1) and (2). Thus, for example, the following theorem holds: if $f(t)$ is a strongly continuous function and $g(t)$ is a function of bounded strong variation ⁽³⁾, then the integral (1) exists.

The set of vector functions integrable in the Bochner sense ⁽³⁾ will be denoted by L . By L_p , $p \geq 1$, we shall mean the set of elements of L whose p -th power of the norm is Lebesgue integrable.

Consider the problem:

Problem A. Find a vector function (control) $f(t) \in L_p$, $p \geq 1$, minimizing (maximizing) the functional

$$I_1(f) = \int_0^1 (x(t), dG(t)), \quad (3)$$

where $x(t)$ is the solution (classical or continuous generalized) of the Cauchy problem

$$\frac{dx}{dt} = A(t)x + f(t), \quad 0 \leq t \leq 1, \quad x(0) = x_0, \quad (4)$$

where $A(t)$, for each fixed $t \in [0, 1]$, is a linear, generally speaking unbounded operator acting in B and having a domain dense in B ; x_0 is a given element of B ; $G(t)$ is a given vector function of bounded strong variation; the derivative dx/dt is understood in the strong sense.

We shall consider Problem A in the class (K) of vector functions $f(t)$ satisfying the constraints:

- a) $0 \leq f(t) \leq F$;
- b)

$$\int_0^1 (f(t), b(t)) dt \leq c,$$

where F is a given element of B ; c is a given nonnegative constant; $b(t)$ is a given vector function with values in B^* and such that

$$b(t) \in L_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

It follows from a) that for every t the value $f(t) \in K$ and $F \in K$.

Problem A was considered in (4) for the special case when B is a one-dimensional Euclidean space.

To solve Problem A, we shall replace it by an equivalent one. Suppose that the operator $A(t)$ satisfies all of Kato's conditions (or part of them) (5). Then the classical (or generalized (6)) solution of the Cauchy problem (4) is determined by the formula

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, s)f(s) ds, \quad (5)$$

where $U(t, s)$ is an operator function of the arguments s, t , $0 \leq s \leq t \leq 1$, strongly continuous in the aggregate of the arguments; for fixed s and t , $U(t, s)$ is a linear bounded operator acting in B , $\|U(t, s)\| \leq 1$.

Substituting now expression (5) into (3), Problem A is reduced to the following:

Problem B. Find $f(t) \in L_p$ from the class (K) which minimizes (maximizes) the functional

$$I(f) = \int_0^1 (f(t), a(t)) dt, \quad (6)$$

where

$$a(t) = \int_t^1 \bar{U}(s, t) dG(s)$$

in the weak topology, and it belongs to L_q ; \bar{U} is the adjoint operator.

Let us note that the maximum problem is reduced to the minimum problem if the sign of the function $a(t)$ is changed. Problem B admits a simple geometric in-

interpretation, if in the plane $\xi O \eta$ one considers the set of points defined by the equalities

$$\xi = \int_0^1 (f(t), a(t)) dt, \quad \eta = \int_0^1 (f(t), b(t)) dt, \quad 0 \leq f(t) \leq F$$

(cf. (4)).

Suppose that $a(t)$ and $b(t)$ satisfy the additional conditions: for any nonpositive k , the functionals $a(t)$ and $kb(t)$, for each fixed $t \in [0, 1]$, are comparable in the cone K .

Then, for the investigation of problem B, we introduce the following sets of points from $[0, 1]$:

$$E^-(k) = [t : a(t) - kb(t) > 0],$$

$$E(k) = [t : a(t) - kb(t) = 0],$$

$$E^+(k) = [t : a(t) - kb(t) < 0].$$

Let $E_0^- = E^-(k_0)$, $E_0 = E(k_0)$, $E_0^+ = E^+(k_0)$, where k_0 is the exact upper bound of the numbers k satisfying the conditions: $k \leq 0$,

$$\int_{E^-} (F, b(t)) dt \leq c.$$

Theorem. If, for sufficiently large $|k|$, $k < 0$, $a(t) - kb(t) \geq 0$ on the cone K for all $t \in [0, 1]$, then the set of minimizing functions $f(t)$ is determined by the relations:

$$f(t) = F \text{ on } E_0^-; \quad f(t) = 0 \text{ on } E_0^+;$$

$f(t)$ is arbitrary on E_0 , satisfying conditions a) and b), if $k_0 = 0$,

and conditions a) and

$$\int_0^1 (f(t), b(t)) dt = c,$$

if $k_0 < 0$.

Proof of the theorem is obtained by estimating from below the difference of the integrals (6), formed for an arbitrary function from the class (K) and the function indicated in the theorem.

A special case of this theorem is the well-known Neyman–Pearson lemma ⁽⁴⁾.

The present theory is applied to the following concrete problem with a strongly self-adjoint and positive definite elliptic operator:

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f(t, \xi), \quad \mathcal{L}u \equiv \sum_{|\alpha| \leq 2m} A\alpha(t, \xi) D^\alpha u, \quad (7)$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ is a point of a finite domain Ω of the n -dimensional real Euclidean space E_n ,

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n,$$

$$D^\alpha = D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_n}, \quad D^{\alpha_j} = \frac{\partial^{\alpha_j}}{\partial \xi_j^{\alpha_j}}, \quad u = (u_1, u_2, \dots, u_N),$$

$A_\alpha(t, \xi)$ are matrix functions, $0 \leq t \leq 1$; the solution of system (7) satisfies the boundary condition $u \in \overset{0}{W}_{2,N}^{(m)}(\Omega)$ and the initial condition $u|_{t=0} = \psi(\xi)$.

A variational problem is posed: let $G(t, \xi)$ and $b(t, \xi)$ be given which, as functions of the argument ξ for fixed t , belong to $\mathcal{L}_{2,N}(\Omega)$; $G(t, \xi)$ is a function of strongly bounded variation with respect to t , and $b(t, \xi) \in L_{q,N}$:

$$\sup \sum_{i=1}^n \left(\int_{\Omega} \sum_{j=1}^N |G_j(t_i, \xi) - G_j(t_{i-1}, \xi)|^2 d\xi \right)^{1/2} < \infty,$$

$$\int_0^1 \left(\int_{\Omega} \sum_{i=1}^N |b_i(t, \xi)|^2 d\xi \right)^{q/2} dt < \infty.$$

Find a (control) $f(t, \xi)$, belonging to $\mathcal{L}_{2,N}(\Omega)$ with respect to ξ and to $L_{p,N}$ with respect to t, ξ , from the class (K) :

$$0 \leq f(t, \xi) \leq \varphi(\xi) \in \mathcal{L}_{2,N}(\Omega), \quad \int_0^1 dt \int_{\Omega} f(t, \xi) b(t, \xi) d\xi \leq c,$$

minimizing the functional

$$I(f) = \int_0^1 \int_{\Omega} u(t, \xi) dG(t, \xi) d\xi.$$

Under additional regularity conditions the problem reduces to variational problem A, if the space B is taken to be the space $\mathcal{L}_{2,N}(\Omega)$, and the cone K is taken to be the set of sequences of nonnegative functions from $\mathcal{L}_{2,N}(\Omega)$.

The theorem given above is applicable to problem (7) if all $b_i(t, \xi) \geq 0$ are bounded above and $a_i(t, \xi)$ are of arbitrary sign but bounded below.

On the basis of the results of P. E. Sobolevskii⁷, the latter problem is considered in the Banach space $\mathcal{L}_p(\Omega)$, $p \geq 1$. Other problems are also considered.

Let us note that questions of optimal control of processes described by partial differential equations have been studied in the literature that has appeared in recent years⁸⁻¹⁰.

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